

On higher derivative gravity with spontaneous symmetry breaking: Hamiltonian analysis of new covariant renormalizable gravity

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Abstract

In order to explore some general features of modified theories of gravity which involve higher derivatives and spontaneous Lorentz and/or diffeomorphism symmetry breaking, we study the recently proposed new version of Covariant Renormalizable Gravity (CRG). CRG is a higher derivative theory of gravity that aims to provide a generally covariant ultraviolet completion of general relativity. Its special features are the presence of projection operators constructed from a constrained scalar field in the action and the spontaneous breaking of Lorentz invariance, which enable the theory to be power-counting renormalizable. We obtain an Arnowitt-Deser-Misner (ADM) representation of the CRG action in 4-dimensional spacetime with respect to a foliation of spacetime adapted to the constrained scalar field. The resulting action is analyzed by using Hamiltonian formalism, which was originally developed for constrained systems by Dirac. It is found that the theory contains two extra propagating degrees of freedom. One is due to the presence of second order time derivatives in the ADM representation of the CRG action. The other is due to the lack of a local Hamiltonian constraint, similarly as in the projectable version of Hořava-Lifshitz gravity. It is argued that CRG contains a degree of freedom that carries negative energy (a ghost) which will destabilize the theory. Such a pathology jeopardizes all higher time derivative field theories where degrees of freedom with positive and negative energies interact with each other, unless the given Lagrangian is appropriately degenerate so that there exist constraints that protect the stability. We conjecture that generally covariant higher derivative theories of gravity which involve spontaneous (constraint induced) Lorentz and/or diffeomorphism symmetry breaking will in general share this problem with CRG.

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1 Introduction

In the recent years modified theories of gravity have attracted a considerable amount of attention. These modifications of General Relativity (GR) aim to improve the behaviour of the theory either at high energies or at large distances. This is motivated by the fact that although GR is very successful in describing gravitational phenomena in intermediate

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distances, it has become increasingly evident that it may need to be completed both at the high energy realm (quantum gravity and renormalizability) and at large distances (dark energy and dark matter) in order to achieve a more plausible theoretical framework and also more convincing agreement with observational data. The fact that according to standard cosmology only four percent of the energy content of the universe has been observed by means other than gravity is a huge gap in our understanding on the nature of gravity.

Perturbative renormalization of GR requires us to include invariants quadratic in curvature into Lagrangian as counterterms [1, 2]. The Riemann tensor squared term can be excluded in 4-dimensional spacetime due to the Gauss-Bonnet topological invariance. Gravitational Lagrangian consisting of the scalar curvature R , scalar curvature squared R^2 and Ricci tensor squared $R_{\mu\nu}R^{\mu\nu}$ terms is indeed renormalizable via dimensional regularization [3]. Unfortunately it includes a massive spin-2 excitation with negative energy, because $R_{\mu\nu}$ includes second-order time derivatives of every component of the metric. This means the theory suffers from Ostrogradskian instability [4], an illness that hampers many higher derivative theories. Therefore much interest has been directed to modified gravity with the Lagrangian $R + \alpha R^2$ [5, 6], which contains only one extra scalar degree of freedom and it is healthy for $\alpha > 0$, although the R^2 term alone is not sufficient for renormalizability. More generally one considers $f(R)$ gravity whose Lagrangian is a non-linear function of the scalar curvature. For a recent review and references, see for example [7]. These theories also have the advantage of being able to realize cosmological phases of accelerated expansion (inflation and present era) without additional dark components. Indeed $f(R)$ gravity has an equivalent representation as a minimally coupled scalar-tensor theory and therefore it is practically equivalent to quintessence in this respect. Other well known examples of modified theories of gravity are for example Brans-Dicke theory [8], another scalar-tensor theory, and the relativistic tensor-vector-scalar implementation [9] of modified Newtonian dynamics [10].

Hořava-Lifshitz (HL) gravity [11] is a power-counting renormalizable field theory of gravity which is based on the idea that space and time scale differently at high energies,

$$\boldsymbol{x} \rightarrow b\boldsymbol{x}, \quad t \rightarrow b^z t, \quad (1.1)$$

with a dynamic critical exponent z . This enables one to modify the ultraviolet behaviour of the graviton propagator to $|\boldsymbol{k}|^{-2z}$, where \boldsymbol{k} is the spatial momentum. In D spatial dimensions, choosing $z = D$ in the ultraviolet fixed point ensures the theory is power-counting renormalizable. Such a spacetime admits a preferred foliation into spatial hypersurfaces, and hence the local Lorentz invariance is broken. Essential extensions of HL gravity have been presented in [12, 13]. Note that more general theories of the HL type have been proposed in Ref. [14].

Recently the so-called Covariant Renormalizable Gravity (CRG) was proposed [16]. CRG aims to provide a power-counting renormalizable field theory of gravity that is covariant under spacetime diffeomorphism and possesses local Lorentz invariance at the fundamental level. CRG aims to achieve a similar ultraviolet behaviour of the graviton propagator as HL gravity, but without introducing explicitly Lorentz noninvariant terms into the action. Lorentz invariance of the graviton propagator of CRG is, however, broken spontaneously at high energies. This is achieved by introducing a scalar field, which is coupled to spacetime in a rather complicated way, and a constraint on the scalar field that breaks Lorentz symmetry spontaneously. Since the CRG action contains higher-order derivatives it exhibits extra degrees of freedom. A new version of CRG has been proposed

[17], where a perturbative analysis around Minkowski spacetime showed that the extra degrees of freedom present in the theory do not propagate. However, we should note that the renormalizability of CRG, as well as of the HL theory, is assumed only based on the power-counting arguments. There are several potential pathologies that could ruin the renormalizability of this theory, such as gradient instabilities, ghosts or strong coupling. Since a violation of Lorentz invariance has never been observed, one could try to argue that CRG is a more natural modification of GR than the explicitly Lorentz noninvariant ones, in particular HL gravity and its generalizations (see e.g. [12–15]). On the other hand Lorentz invariance could equally well be broken explicitly at very high energies as long as it is somehow restored at sufficiently low energies.

At present most modified theories of gravity should be treated as effective or phenomenological theories, since they are not derived from any compelling first principles, rather constructed to meet some specific purposes. Nevertheless these theories can teach us a great deal about the aspects of gravity, and ultimately help us in laying down the foundations for the next paradigm of space, time and gravity.

Hamiltonian formalism provides a powerful tool for the analysis of constrained systems such as gravity. For example, it has been shown by Hamiltonian analysis that HL gravity can be physically consistent ($N > 0$) at high energies only if either the projectability condition is imposed on the lapse function N [18] or the potential part of the action is extended with terms that contain spatial derivatives of N [19, 20] as the vector

$$a_i = \frac{\partial_i N}{N}. \quad (1.2)$$

A similar result regarding the projectability of N has also been obtained for the more general modified $F(R)$ HL gravity [21]. The latter approach is the so-called consistent extension of HL gravity [13], where the extra degree of freedom present in the theory [22] has a healthy quadratic action due to extension of the action by the a_i terms. In a previous paper [23] we studied the first version of CRG from the point of view of Hamiltonian analysis. It was found that there indeed exist extra degrees of freedom compared to GR, because there are not enough constraints to eliminate them. In this paper we analyze the new improved version of CRG, concentrating on the renormalizable model ($z = 3$) in 4-dimensional spacetime. In particular we are interested in whether the new model contains extra modes similarly as original CRG and whether the possible extra modes are pathological ghosts. We shall see that the results of this study will be of interest to other conceivable generally covariant higher derivative theories of gravity which involve spontaneous (constraint induced) Lorentz and/or diffeomorphism symmetry breaking.

First the action of new CRG is introduced in Sec. 2. We obtain the Arnowitt-Deser-Misner (ADM) representation of the new CRG action with respect to a foliation of spacetime adapted to the constrained scalar field in Sec. 3. The resulting action contains time derivative of the extrinsic curvature and powers of the extrinsic curvature up to sixth order. In Sec. 4, introducing some additional fields enables us to obtain a first-order action with a kinetic part quadratic in extrinsic curvature and other first time derivatives. Certain solutions of the first-order action are discussed in Sec. 5. In Sec. 6, we analyze the action using Hamiltonian formalism. A few alternative sets of variables for Hamiltonian formulation are considered in Sec. 7. Conclusions and some further discussion are presented in Sec. 8.

2 Action

We consider the new version of covariant renormalizable gravity with projectors [17]. For definiteness we shall consider the specific model corresponding to critical exponent $z = 3$ which should be power-counting renormalizable in 4-dimensional spacetime. The action reads

$$S_3 = \int d^4x \sqrt{-^{(4)}g} \left[\frac{^{(4)}R}{2\kappa^2} - \alpha P_\alpha{}^\mu P_\beta{}^\nu \left(^{(4)}R_{\mu\nu} - \frac{1}{2U_0} \partial_\rho \phi \nabla^\rho \nabla_\mu \nabla_\nu \phi \right) \right. \\ \left. \times (\partial^\mu \phi \partial^\nu \phi \nabla_\mu \nabla_\nu - \partial_\mu \phi \partial^\mu \phi \nabla^\nu \nabla_\nu) P^{\alpha\mu} P^{\beta\nu} \left(^{(4)}R_{\mu\nu} - \frac{1}{2U_0} \partial_\rho \phi \nabla^\rho \nabla_\mu \nabla_\nu \phi \right) \right. \\ \left. - \lambda \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U_0 \right) \right], \quad (2.1)$$

where the projector is defined by

$$P_\alpha{}^\mu = \delta_\alpha^\mu + \frac{\partial_\alpha \phi \partial^\mu \phi}{2U_0}. \quad (2.2)$$

Variation of the action (2.1) with respect to the Lagrange multiplier λ implies the constraint on the scalar field ϕ as

$$\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U_0 = 0. \quad (2.3)$$

For simplicity we consider U_0 to be a positive constant, as was considered in the original proposal [17], but more generally it could be any positive function of ϕ .

Choosing a solution of the equation of motion (2.3) spontaneously breaks the Lorentz invariance and/or the full general covariance of the action (2.1) [17]. According to the constraint (2.3) $\partial^\mu \phi$ is timelike and hence one can choose the direction of time to be parallel to $\partial_\mu \phi$ at least locally. Then one finds that the modification added to Einstein-Hilbert action in (2.1), i.e., the term with the coupling constant α , turns out to contain only spatial derivatives of the perturbation $h_{\mu\nu}$ to Minkowski metric: $^{(4)}g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the present case ($z = 3$) there are six spatial derivatives of $h_{\mu\nu}$, while in the general case there are $2z$ spatial derivatives, yielding the desired $|\mathbf{k}|^{-2z}$ modification of the graviton propagator. The form of this modification to GR was specifically constructed so that time derivatives of $h_{\mu\nu}$ cancel out in the action once the Lorentz symmetry is spontaneously broken and the direction of time has been fixed. This means that all the problems of higher time derivative theories can be avoided in this linearized formulation, quite similarly as in HL gravity where higher time derivatives are explicitly excluded in the definition of action.

Although we concentrate on the renormalizable case $z = 3$ in four dimensions, actions corresponding to other (higher) values z could be considered in a similar fashion by using the results we will obtain in the following sections.

3 Arnowitt-Deser-Misner representation

We consider the ADM decomposition of the gravitational field [24]; for reviews and mathematical background, see [25]. Spacetime is assumed to admit a foliation into spacelike hypersurfaces Σ_t of constant t . The metric of spacetime can be written as

$$^{(4)}g_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \quad (3.1)$$

where $g_{\mu\nu}$ is the induced metric on the spacelike hypersurfaces Σ_t and n_μ is the future-directed unit normal to Σ_t . The ADM variables consists of the lapse function N , the shift vector N^i and the spatial metric g_{ij} ($i, j = 1, 2, 3$). The unit normal to Σ_t can be written in terms of the ADM variables as

$$n_\mu = -N\nabla_\mu t = (-N, \mathbf{0}), \quad n^\mu = (n^0, n^i) = \left(\frac{1}{N}, -\frac{N^i}{N}\right). \quad (3.2)$$

Thus the ADM representation for the metric of spacetime reads

$${}^{(4)}g_{00} = -N^2 + N_i N^i, \quad {}^{(4)}g_{0i} = {}^{(4)}g_{i0} = N_i, \quad {}^{(4)}g_{ij} = g_{ij}, \quad (3.3)$$

where $N_i = g_{ij}N^j$. Contravariant components of the metric of spacetime are

$${}^{(4)}g^{00} = -\frac{1}{N^2}, \quad {}^{(4)}g^{0i} = {}^{(4)}g^{i0} = \frac{N^i}{N^2}, \quad {}^{(4)}g^{ij} = g^{ij} - \frac{N^i N^j}{N^2}, \quad (3.4)$$

where $g^{ij}g_{jk} = \delta_k^i$. The extrinsic curvature of the spatial hypersurface Σ_t is defined by

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - 2D_{(i}N_{j)}) , \quad K = g^{ij}K_{ij}, \quad (3.5)$$

where the dot denotes the derivative with respect to time t . Quantities defined on the spacetime \mathcal{M} and associated with its metric ${}^{(4)}g_{\mu\nu}$ are marked with the prefix ${}^{(4)}$. We denote the covariant derivatives on \mathcal{M} and Σ_t by ∇ and D , respectively. For further details on the notation, one can also see [23].

3.1 Space-time decomposition of the action

Instead of deriving the general Arnowitt-Deser-Misner (ADM) representation of the action (2.1), as was first done in our analysis of the original formulation of CRG [23], we directly consider a specially chosen foliation of spacetime. The constraint (2.3) on ϕ ensures that the vector $\partial^\mu \phi$ is timelike everywhere, when $U_0 > 0$ is assumed. Therefore there exists a preferred foliation of spacetime into spatial hypersurfaces Σ_t whose unit normal is given by

$$n^\mu = -\frac{\partial^\mu \phi}{\sqrt{-\partial_\nu \phi \partial^\nu \phi}} = -\frac{\partial^\mu \phi}{\sqrt{2U_0}}. \quad (3.6)$$

Then the constraint (2.3) reduces to the condition that the normal has unit norm

$$n_\mu n^\mu = -1, \quad (3.7)$$

which is presumed for the unit normal by definition. Hence we are effectively substituting the solution of the equation of motion (2.3) back into the action. The projector (2.2) becomes the orthogonal projector onto the spatial hypersurface Σ_t , $P_\alpha{}^\mu = g^\mu{}_\alpha$, which is defined as

$$g^\mu{}_\alpha = \delta^\mu{}_\alpha + n^\mu n_\alpha. \quad (3.8)$$

Now we can write

$$\partial^\mu \phi = -\sqrt{2U_0}n^\mu, \quad \partial_\mu \phi = -\sqrt{2U_0}n_\mu. \quad (3.9)$$

From (3.2) and (3.9) we see that in this foliation ϕ is constant on Σ_t , $\phi = \phi(t)$. Clearly this construction bears some similarity to the Stückelberg formalism used in HL gravity [22, 26], where the foliation structure of spacetime is encoded into a scalar field in order to

achieve a manifestly covariant action and “transfer” the extra degree of freedom from the metric to the scalar. Here the scalar field ϕ is present from the beginning and we choose to work with a foliation of spacetime defined by ϕ via (3.6). Indeed, one could choose to decompose the action with respect to an arbitrary foliation of spacetime, with no relation to ϕ , but in that case the action turns out quite complicated, involving higher-order time derivatives up to fifth order. Note that ϕ is no longer an independent variable, but rather related to U_0 and the lapse N . The constraint (2.3) on ϕ reduces to

$$-\frac{\dot{\phi}^2}{2N^2} + U_0 = 0. \quad (3.10)$$

and we can integrate it for the scalar ϕ

$$\phi(t) = \phi(t_0) + \sqrt{2U_0} \int_{t_0}^t dt' N(t'). \quad (3.11)$$

Evidently the choice $N = 1/\sqrt{2U_0}$ corresponds to actually choosing the scalar field as the time coordinate, $\phi = t$. The relation (3.10) or (3.11) implies that the lapse N must be constant on Σ_t too, $N = N(t)$, because otherwise ϕ could not be constant on Σ_t . In order to preserve the conditions $\phi = \phi(t)$ and $N = N(t)$ we restrict the symmetry under diffeomorphisms of spacetime to the symmetry under foliation-preserving diffeomorphisms, given in the infinitesimal form as

$$\delta t = f(t), \quad \delta \mathbf{x} = \boldsymbol{\xi}(t, \mathbf{x}). \quad (3.12)$$

This is the main symmetry group of the Hořava-Lifshitz (HL) gravity. In the language of Hořava’s theory we would say that both ϕ and N are projectable — like the lapse is in the projectable version of HL gravity. Here we consider (3.12) as a partial gauge fixing of the diffeomorphism symmetry, which is required by the choice of working with the preferred foliation.

Now the action (2.1) can be written

$$S_3 = \int d^4x \sqrt{-^{(4)}g} \left[\frac{{}^{(4)}R}{2\kappa^2} - \alpha g^\mu{}_\alpha g^\nu{}_\beta ({}^{(4)}R_{\mu\nu} - \nabla_n K_{\mu\nu} + a_\mu a_\nu) \right. \\ \left. \times 2U_0 (n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu) g^{\mu\alpha} g^{\nu\beta} ({}^{(4)}R_{\mu\nu} - \nabla_n K_{\mu\nu} + a_\mu a_\nu) \right], \quad (3.13)$$

where we denote $\nabla_n = n^\mu \nabla_\mu$ and use (3.9) as $\nabla_\mu \phi = -\sqrt{2U_0} n_\mu$ and apply the following geometrical identities

$$\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu, \quad (3.14)$$

$$a_\mu = \nabla_n n_\mu = D_\mu \ln N, \quad (3.15)$$

in order to write

$$\frac{1}{2U_0} \partial_\rho \phi \nabla^\rho \nabla_\mu \nabla_\nu \phi = \nabla_n K_{\mu\nu} - a_\mu a_\nu - n_\mu \nabla_n a_\nu. \quad (3.16)$$

The vector a_μ has the physical interpretation of being the acceleration of an observer with 4-velocity n^μ . It is always orthogonal to n^μ , $a_\mu n^\mu = 0$, and hence tangent to Σ_t , $g^\mu{}_\alpha a_\mu = a_\alpha$.

In the action (3.13), we recognize $g^\mu_\alpha g^\nu_\beta {}^{(4)}R_{\mu\nu}$ as the component of the Ricci tensor of spacetime that is tangent to Σ_t . With the help of the Gauss relation (A.1) and the Ricci equation (A.3) it can be written

$$g^\mu_\alpha g^\nu_\beta {}^{(4)}R_{\mu\nu} = R_{\alpha\beta} + K K_{\alpha\beta} - 2K_{\alpha\mu} K^\mu_\beta - \frac{1}{N} D_\alpha D_\beta N + \frac{1}{N} \mathcal{L}_{Nn} K_{\alpha\beta}, \quad (3.17)$$

where $R_{\alpha\beta}$ is the Ricci tensor of the hypersurface Σ_t and \mathcal{L}_{Nn} denotes the Lie derivative along the 4-vector $Nn^\mu = (1, -N^i)$. Then we calculate the Lie derivative in (3.17)

$$\frac{1}{N} \mathcal{L}_{Nn} K_{\alpha\beta} = \nabla_n K_{\alpha\beta} + (K_\alpha^\mu - n_\alpha a^\mu) K_{\mu\beta} + (K_\beta^\mu - n_\beta a^\mu) K_{\alpha\mu}. \quad (3.18)$$

For any tensor T that is tangent to Σ_t its Lie derivative $\mathcal{L}_{Nn} T$ is also tangent to Σ_t . This follows from $\mathcal{L}_{Nn} g^\mu_\alpha = 0$. Thus we can project both sides of (3.18) onto Σ_t and obtain

$$\frac{1}{N} \mathcal{L}_{Nn} K_{\alpha\beta} = g^\mu_\alpha g^\nu_\beta \nabla_n K_{\mu\nu} + 2K_{\alpha\mu} K^\mu_\beta. \quad (3.19)$$

Substituting (3.19) into (3.17) enables us to write

$$g^\mu_\alpha g^\nu_\beta {}^{(4)}R_{\mu\nu} = R_{\alpha\beta} + K K_{\alpha\beta} - \frac{1}{N} D_\alpha D_\beta N + g^\mu_\alpha g^\nu_\beta \nabla_n K_{\mu\nu}. \quad (3.20)$$

Thus in the action (3.13) we obtain the decomposition

$$g^\mu_\alpha g^\nu_\beta ({}^{(4)}R_{\mu\nu} - \nabla_n K_{\mu\nu} + a_\mu a_\nu) = R_{\alpha\beta} + K K_{\alpha\beta} - D_\alpha a_\beta, \quad (3.21)$$

where we can actually drop the last term in the right-hand side, $-D_\alpha a_\beta = a_\alpha a_\beta - \frac{1}{N} D_\alpha D_\beta N$, since the condition $N = N(t)$ implies that a_μ vanishes

$$a_\mu = N^{-1} D_\mu N = N^{-1} g^\nu_\mu \partial_\nu N = 0, \quad (3.22)$$

because $g^0_\mu = 0$ and $\partial_i N = 0$. All the terms in the right-hand side of (3.21) are tangent to Σ_t , which is the essential geometrical implication for introducing the projectors into the action of new CRG.

Using (3.21) we can write the action (3.13) as

$$S_3 = \int d^4x \sqrt{-{}^{(4)}g} \left[\frac{{}^{(4)}R}{2\kappa^2} - 2\alpha U_0 (R^{\alpha\beta} + K K^{\alpha\beta}) (n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu) (R_{\alpha\beta} + K K_{\alpha\beta}) \right]. \quad (3.23)$$

What remains to be decomposed are the covariant derivatives of tensor fields that are tangent to the spatial hypersurfaces Σ_t , namely

$$\nabla_\mu \nabla_\nu (R_{\alpha\beta} + K K_{\alpha\beta}). \quad (3.24)$$

In the original formulation of CRG one instead takes covariant derivatives of a scalar, namely the component of the Einstein tensor that is purely normal to Σ_t , i.e. $\partial^\mu \phi \partial^\nu \phi {}^{(4)}G_{\mu\nu} \propto n^\mu n^\nu {}^{(4)}G_{\mu\nu} = \frac{1}{2} (K^2 - K_{ij} K^{ij} + R)$, which are much easier to decompose. The spatial covariant derivative D of a (k, l) -tensor field T on Σ_t is given in terms of the covariant derivative ∇ of spacetime as

$$D_\mu T^{\nu_1 \dots \nu_k}_{\rho_1 \dots \rho_l} = g^\sigma_\mu g^{\nu_1}_{\alpha_1} \dots g^{\nu_k}_{\alpha_k} g^{\beta_1}_{\rho_1} \dots g^{\beta_l}_{\rho_l} \nabla_\sigma T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l}, \quad (3.25)$$

where in the right-hand side one considers the extension of T on spacetime. This relation of D and ∇ can be used in the decomposition of covariant derivatives of tensor fields tangent to Σ_t . In the Appendix A.3, we present the calculation of space-time decomposition of first and second order covariant derivatives of a symmetric tensor field $A_{\alpha\beta}$ that is tangent to Σ_t .

Here we consider the specific combination of second order covariant derivatives that appears in the action (3.23):

$$\begin{aligned}
(n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu) A_{\alpha\beta} &= g^{\nu\mu} g^\rho_\mu g^\sigma_\alpha g^\lambda_\beta \nabla_\nu \nabla_\rho A_{\sigma\lambda} - n_\alpha g^{\nu\mu} g^\rho_\mu n^\sigma g^\lambda_\beta \nabla_\nu \nabla_\rho A_{\sigma\lambda} \\
&\quad - n_\beta g^{\nu\mu} g^\rho_\mu g^\sigma_\alpha n^\lambda \nabla_\nu \nabla_\rho A_{\sigma\lambda} + n_\alpha n_\beta g^{\nu\mu} g^\rho_\mu n^\sigma n^\lambda \nabla_\nu \nabla_\rho A_{\sigma\lambda} \\
&= D^\mu D_\mu A_{\alpha\beta} - K \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\mu A_{\beta)\mu} \right) \\
&\quad + 2K^{\mu\nu} K_{\mu(\alpha} A_{\nu|\beta)} \\
&\quad + n_\alpha (2K^{\mu\nu} D_\mu A_{\nu\beta} + D^\mu K_{\mu\nu} A^\nu_\beta - K a^\mu A_{\mu\beta}) \\
&\quad + n_\beta (2K^{\mu\nu} D_\mu A_{\alpha\nu} + D^\mu K_{\mu\nu} A_\alpha{}^\nu - K A_{\alpha\mu} a^\mu) \\
&\quad + n_\alpha n_\beta 2K^{\mu\nu} K_\mu{}^\rho A_{\nu\rho}.
\end{aligned} \tag{3.26}$$

When we multiply (3.26) with another symmetric tensor field $B^{\alpha\beta}$ that is tangent to Σ_t , we obtain

$$\begin{aligned}
B^{\alpha\beta} (n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu) A_{\alpha\beta} &= B^{ij} \left(D^k D_k A_{ij} - K \frac{1}{N} \mathcal{L}_{Nn} A_{ij} \right. \\
&\quad \left. + 2K K_i{}^k A_{kj} + 2K_{ik} K^{kl} A_{lj} \right).
\end{aligned} \tag{3.27}$$

In the right-hand side of (3.27) all the tensor fields are tangent to Σ_t , which enabled us to write the contractions over spatial components. Substituting

$$A_{ij} = R_{ij} + K K_{ij}, \quad B^{ij} = R^{ij} + K K^{ij} \tag{3.28}$$

into (3.27) gives the ADM representation of the action (3.13)

$$\begin{aligned}
S_3 &= \int d^4x \sqrt{g} N \left\{ \frac{K_{ij} K^{ij} - K^2 + R}{2\kappa^2} - 2\alpha U_0 (R^{ij} + K K^{ij}) \left[D^k D_k (R_{ij} + K K_{ij}) \right. \right. \\
&\quad \left. \left. - K \frac{1}{N} \mathcal{L}_{Nn} (R_{ij} + K K_{ij}) + 2K K_i{}^k (R_{kj} + K K_{kj}) + 2K_{ik} K^{kl} (R_{lj} + K K_{lj}) \right] \right\}.
\end{aligned} \tag{3.29}$$

Here we have also included the usual decomposition of the scalar curvature of spacetime (A.4) and dropped the divergence terms; we assume the spatial manifold is compact and that it has no boundary. The first thing to note is that the Lagrangian of the new CRG given in (3.29) contains kinetic terms involving the extrinsic curvature to the sixth power, while conventional gravitational Lagrangians only include kinetic terms quadratic in the extrinsic curvature. The action also contains second order time derivatives of the metric in the term involving the Lie derivative of the extrinsic curvature. Namely in the Lagrangian we have a term that contains

$$\begin{aligned}
K (R^{ij} + K K^{ij}) \mathcal{L}_{Nn} (R_{ij} + K K_{ij}) &= K (R^{ij} + K K^{ij}) [\partial_t (R_{ij} + K K_{ij}) \\
&\quad - \mathcal{L}_N (R_{ij} + K K_{ij})],
\end{aligned} \tag{3.30}$$

where \mathcal{L}_N denotes the Lie derivative along the shift vector N^i in Σ_t . Space-time decomposition of the Lie derivative \mathcal{L}_{N_n} of a covariant tensor tangent to Σ_t can be found in Appendix A.3.

Recall that in the linearized treatment [17] the part of the action with the coupling constant α contained only spatial derivatives of the perturbation to Minkowski metric. Here we obtain that our ADM representation of the same part of the action contains first and second order time derivatives. Therefore already at this point we can expect to obtain some results that will differ from [17], in particular due to the presence of the higher time derivative term (3.30).

ADM representations of CRG actions corresponding to values of the critical exponent z other than $z = 3$ can be written by using the results (3.21) and (3.26). It is found that in general such actions contain kinetic terms involving the extrinsic curvature and possibly its time derivatives to the $2z$ -th power. Higher-order time derivatives are present because every instance of the derivative operator decomposed in (3.26) adds one more time derivative. The only exception is the nonrenormalizable case $z = 2$, where the action is like (3.23) but without the derivative operator decomposed in (3.26), which involves only first order time derivatives albeit its kinetic part is still modified substantially. We will not present the ADM representations of the actions for higher z here since generalization of the present results is straightforward.

ADM representations of other conceivable generally covariant higher derivative theories of gravity which aim to be power-counting renormalizable by involving spontaneous (constraint induced) Lorentz and/or diffeomorphism symmetry breaking will likely share some characteristics with CRG. In particular, it is unlikely that all higher time derivatives could be cancelled by such construction. However, constraints that also contain higher time derivatives might just be able to serve such purpose.

4 First-order Lagrangian for Hamiltonian formalism

A Hamiltonian formulation for higher derivative theories with regular Lagrangians was first developed by Ostrogradski [27]. Some decades after Dirac developed Hamiltonian formalism for constrained systems [28, 29] it was generalized to higher derivative theories [30–32]. Soon Hamiltonian formulations of higher derivative theories of gravity were constructed for the first times [33–35]. Hamiltonian formulation of actions that involve higher-order time derivatives requires one to introduce a pair of new independent variables for each higher-order time derivative of a variable. There generally exist several different choices of such additional variables, which each yield a different Hamiltonian formulation of a given higher-order action. Such Hamiltonian formulations are connected by canonical transformations [35] and hence they are classically equivalent. But those canonical transformation can be highly nonlinear. Thus there is no guarantee that the Hamiltonian formulations remain equivalent after quantization.

Regular Lagrangians that depend on higher-order time derivatives of dynamical variables are known to possess degrees of freedom which carry both negative and positive energies [27]. In an interacting higher derivative field theory such ghosts necessarily destabilize the theory when the Lagrangian is regular (nondegenerate), since any state of the system can and will further decay into excitations with compensating negative and positive energies; for an example, see [36]. Higher derivative theories whose Lagrangian are singular (degenerate) can sometimes avoid the Ostrogradskian instability. Theories which possesses continuous symmetries are always degenerate, in particular gauge theories

such as gravity, and hence they have a chance to avoid the instability. Thus for higher derivative theories of gravity, the existence and behaviour of ghosts has to be checked in each theory. The same applies to other pathologies such as strong coupling of extra degrees of freedom.

A relatively simple example of a higher derivative theory of gravity is provided by $f(^{(4)}R)$ gravity, whose Lagrangian is an arbitrary function of the scalar curvature $^{(4)}R$ of spacetime. The scalar curvature is second order in time derivatives (A.4). As a result the theory contains an extra degree of freedom compared to GR, whose Einstein-Hilbert Lagrangian is just $^{(4)}R$. For a recent review of $f(^{(4)}R)$ gravity, see for example [7]. $f(^{(4)}R)$ gravity has several known Hamiltonian formulations based on different choices for the higher-order variable associated with the extra degree of freedom. One can regard the scalar curvature of spacetime as the additional dynamical variable [34]. Since the second order time derivative in the scalar curvature (A.4) appears as \dot{K} , we can follow the approach of [35] by regarding the trace of the extrinsic curvature as an alternative scalar variable. A third alternative is provided by the fact that $f(^{(4)}R)$ gravity is equivalent to a scalar-tensor theory where the scalar is minimally coupled to GR. The extra scalar degree of freedom in $f(^{(4)}R)$ gravity is indeed stable if the potential is well behaved. For a discussion of the Ostrogradskian instability and the reason why $f(^{(4)}R)$ gravity is able to avoid it, see [4]. For a recent review of the aforementioned three Hamiltonian formulations of $f(^{(4)}R)$ gravity, see [37].

In the action (3.29), there is a term (3.30) that contains a time derivative of each component of the extrinsic curvature K_{ij} . That is a second order time derivative of each component of the metric. Therefore we shall introduce two new symmetric rank 2 tensor variables. The way in which these new variables are defined is a matter choice. Since our action contains the second order time derivatives in \dot{K} and \dot{K}_{ij} it would be natural to follow the approach originated in [35] where K_{ij} are taken as additional independent variables. We, however, choose alternative variables that simplify the action further. In order to obtain an action whose Lagrangian is quadratic in the extrinsic curvature and the first time derivatives of the additional higher-order variables we shall introduce an additional independent symmetric tensor variable ζ_{ij} , which is related to the metric variables by

$$\zeta_{ij} = R_{ij} + K K_{ij} \quad (4.1)$$

This choice also enables us to get rid of terms that involve the time derivative \dot{R}_{ij} of the spatial Ricci tensor. In order to enforce the relation (4.1) another symmetric tensor field ξ^{ij} will be introduced as a Lagrange multiplier. We replace the action (3.29) by

$$S_3 = \int d^4x \sqrt{g} N \left[\frac{K_{ij} K^{ij} - K^2 + R}{2\kappa^2} - 2\alpha U_0 g^{ik} g^{jl} \zeta_{kl} \left(D^k D_k \zeta_{ij} - K \frac{1}{N} \mathcal{L}_{Nn} \zeta_{ij} \right. \right. \\ \left. \left. + 2K K_i^k \zeta_{kj} + 2K_{ik} K^{kl} \zeta_{lj} \right) - \alpha \xi^{ij} (\zeta_{ij} - R_{ij} - K K_{ij}) \right]. \quad (4.2)$$

Variation of the action with respect to ξ^{ij} gives the equation (4.1). Substituting it back into the action would give the original action (3.29). Since the action does not contain any time derivatives of ξ^{ij} these variables have an auxiliary character; they are not expected to carry any physical degrees of freedom. In the action (4.2), the term that contains a time derivative of ζ_{ij} can be written as

$$2\alpha U_0 K g^{ik} g^{jl} \zeta_{kl} \mathcal{L}_{Nn} \zeta_{ij} = 2\alpha U_0 K g^{ik} g^{jl} \zeta_{kl} \left(\dot{\zeta}_{ij} - \mathcal{L}_N \zeta_{ij} \right), \quad (4.3)$$

where \mathcal{L}_N denotes the Lie derivative along the shift vector N^i in Σ_t . Thus at least some of the variables ζ_{ij} are expected to carry propagating degrees of freedom.

In the actions (3.29) and (4.2) N had to be restricted to a function of time only as was discussed in Sec. 3. In these actions, it would be possible to promote the lapse function to possess dependence on both space and time, since the symmetry under (3.12) would be retained. But the resulting action might possess very different properties compared to the ADM representation we have here derived from the CRG action (2.1). Hamiltonian structure of such a generalized action would certainly bare some differences, in particular the existence of a local Hamiltonian constraint, and its study would be interesting. Here we shall, however, constrain ourselves to the study of the action (4.2) with the projectability of N implied by the constraint on the scalar field (3.10).

Two alternative sets of variables for Hamiltonian formulation of new CRG are discussed in Sec. 7.

5 On solutions for the first-order Lagrangian

Let us consider solutions for the first-order Lagrangian, i.e., configurations for which the action (4.2) is extremal. When the Lagrange multiplier fields ξ^{ij} are set to zero, the equations of motion obtained by varying the higher-order variables ζ_{ij} are linear and homogeneous in ζ_{ij} . They have a static solution $\zeta_{ij} = 0$ provided that the initial value for ζ_{ij} is zero everywhere on the $t = 0$ hypersurface Σ_0 . Then the equations of motion (4.1) obtained by varying ξ^{ij} give

$$R_{ij} + K K_{ij} = 0, \quad (5.1)$$

which imply that the scalar intrinsic and extrinsic curvature are related by $R = -K^2$. Substituting these solutions back into the action gives

$$S = \int d^4x \sqrt{g} N \frac{K_{ij} K^{ij} - 2K^2}{2\kappa^2} \quad (5.2)$$

with the intrinsic and extrinsic curvature related by (5.1). This kind of action with no curvature dependent terms in the potential is called ultralocal gravity. Such an ultralocal theory of gravity where the kinetic part of the action is identical to that of GR and the potential consists only of the cosmological constant was originally proposed in [38]. The Hamiltonian of ultralocal gravity has the same form as in GR, written in terms of the ADM variables as

$$H = \int d^3\mathbf{x} (N\mathcal{H}_0 + N^i\mathcal{H}_i), \quad (5.3)$$

where \mathcal{H}_0 is the Hamiltonian constraint and \mathcal{H}_i is the momentum constraint. The appealing characteristic of ultralocal gravity is that the algebra of constraints is a true Lie algebra, because the Poisson bracket of Hamiltonian constraints $\mathcal{H}_0(\mathbf{x})$ and $\mathcal{H}_0(\mathbf{y})$ is zero. In GR the Poisson bracket of Hamiltonian constraints is equal to a linear combination of momentum constraints $\mathcal{H}_i(\mathbf{x})$ with a field-dependent “structure constraint”. Thus the constraints of GR lack Lie algebra structure. The momentum constraint of ultralocal is identical to that of GR and hence the rest of the constraint algebra of ultralocal gravity coincide with GR.

Note that the kinetic part of the action (5.2) is not identical to GR since its second term is modified by the factor 2. Rather this action formally corresponds to the ultralocal case of HL gravity [11] in the limit $z \rightarrow 0$, $\lambda \rightarrow 2$ and with vanishing cosmological constant.

Here λ is the coupling modifying the kinetic part of the Lagrangian: $K_{ij}K^{ij} - \lambda K^2$. The only difference is that here the spatial Ricci tensor is given in terms of extrinsic curvature by (5.1). A special case of this kind of solution is static flat spacetime, so that $R_{ij} = 0$ and $K_{ij} = 0$ satisfy (5.1). Thus we confirm that CRG action has a static flat solution. However, this does not necessarily mean that it is a stable vacuum state.

When the initial geometry of spacetime deviates from (5.1) even slightly, the dynamics of spacetime produced by the action (4.2) is very different compared to the case discussed above. The equations of motion obtained by varying g^{ij} and ζ_{ij} are highly nontrivial. Indeed the dynamics of the metric and the higher-order variables ζ_{ij} is very complicated. Thus we do not attempt to solve the equations of motion in the general case. Instead we seek to understand the system through Hamiltonian analysis.

6 Hamiltonian analysis

Let us analyze the new CRG action (4.2) by using the Hamiltonian formalism generalized for constrained systems [28, 29]. For reviews, see [39].

In Sec. 5, we saw that when $\zeta_{ij} = 0$, i.e., when (5.1) holds, the action reduces to an ultralocal special case of HL gravity (5.2). Now we aim to understand the structure and dynamics of the theory when ζ_{ij} is nonzero. We assume that at least two components of ζ_{ij} are nonzero, while the rest of the components can attain any values. In principle it makes no difference which of the components are chosen to be nonzero. We choose one of the nonvanishing components to be the trace of ζ_{ij} due to notational elegance.

First we define the canonical momenta. Since the action (4.2) is independent of the time derivatives of N , N^i and ξ^{ij} , their canonically conjugated momenta, p_N , p_i and p_{ij}^ξ , respectively, are primary constraints:

$$p_N \approx 0, \quad p_i(\mathbf{x}) \approx 0, \quad p_{ij}^\xi(\mathbf{x}) \approx 0. \quad (6.1)$$

The momenta canonically conjugate to g_{ij} and ζ_{ij} are defined by

$$\begin{aligned} p^{ij} = \frac{\delta S_3}{\delta \dot{g}_{ij}} = \sqrt{g} \left[\frac{K^{ij} - g^{ij}K}{2\kappa^2} + \frac{\alpha U_0}{N} g^{ij} \left(\dot{\zeta}_{kl} - \mathcal{L}_N \zeta_{kl} \right) g^{km} g^{ln} \zeta_{mn} \right. \\ \left. - 2\alpha U_0 \left(g^{ij} K^{kl} + g^{ik} K^{jl} + g^{jk} K^{il} + g^{il} K^{jk} \right) \zeta_{km} \zeta_{ln} g^{mn} \right. \\ \left. + \frac{\alpha}{2} \left(g^{ij} \xi^{kl} K_{kl} + \xi^{ij} K \right) \right], \end{aligned} \quad (6.2)$$

$$p_\zeta^{ij} = \frac{\delta S_3}{\delta \dot{\zeta}_{ij}} = \sqrt{g} 2\alpha U_0 K g^{ik} g^{jl} \zeta_{kl}. \quad (6.3)$$

We shall adopt a convention where the trace component of a tensor or a tensor density is denoted without indices and the traceless component is denoted with the bar accent. For instance according to this convention we decompose the variables ζ_{ij} and p_ζ^{ij} as

$$\zeta_{ij} = \bar{\zeta}_{ij} + \frac{1}{3} g_{ij} \zeta, \quad p_\zeta^{ij} = \bar{p}_\zeta^{ij} + \frac{1}{3} g_{ij} p_\zeta, \quad (6.4)$$

where $\bar{\zeta}_{ij}$ and \bar{p}_ζ^{ij} are the traceless components and $\zeta = g^{ij} \zeta_{ij}$ and $p_\zeta = g_{ij} p_\zeta^{ij}$ are the trace components. The same notation can be used for the other variables and tensors in

general. Taking the traces of the momenta (6.2) and (6.3) gives

$$p = \sqrt{g} \left[-\frac{K}{\kappa^2} + \frac{3\alpha U_0}{N} \left(\dot{\zeta}_{ij} - \mathcal{L}_N \zeta_{ij} \right) g^{ik} g^{jl} \zeta_{kl} - \alpha U_0 (10K^{ij} + 2K g^{ij}) \zeta_{ik} \zeta_{jl} g^{kl} + \frac{\alpha}{2} (3\xi^{ij} K_{ij} + \xi K) \right], \quad (6.5)$$

$$p_\zeta = \sqrt{g} 2\alpha U_0 K \zeta. \quad (6.6)$$

where we use the aforementioned convention to denote $p = g_{ij} p^{ij}$, $\xi = g_{ij} \xi^{ij}$ etc. Assuming ζ_{ij} has a nonvanishing trace, $\zeta \neq 0$, and the coupling constants $\alpha \neq 0$ and $U_0 > 0$, we solve (6.6) for the trace of the extrinsic curvature

$$K = \frac{1}{\sqrt{g}} \frac{p_\zeta}{2\alpha U_0 \zeta} \quad (6.7)$$

Additional primary constraints can be obtained by substituting (6.7) back into (6.3). The trace of the resulting equation is a trivial identity, but its traceless part expresses \bar{p}_ζ^{ij} in terms of other variables. Thus we obtain five new primary constraints

$$\bar{\Pi}^{ij} = \bar{p}_\zeta^{ij} - g^{ik} g^{jl} \bar{\zeta}_{kl} \frac{p_\zeta}{\zeta} \approx 0, \quad (6.8)$$

which form a symmetric traceless tensor density. This primary constraint tells us that only the trace component of the momenta p_ζ^{ij} is an independent variable. In order to solve the traceless component \bar{K}_{ij} of the extrinsic curvature we separate it from the already solved trace component (6.7) as

$$K_{ij} = \bar{K}_{ij} + \frac{1}{3} g_{ij} K = \bar{K}_{ij} + \frac{1}{\sqrt{g}} \frac{g_{ij}}{6\alpha U_0} \frac{p_\zeta}{\zeta}. \quad (6.9)$$

Then we substitute (6.7) and (6.9) into (6.2) and (6.5). The resulting equations can be solved for \bar{K}_{ij} . We obtain the equation

$$p^{ij} - \frac{1}{3} g^{ij} p + \frac{5}{3} \left(g^{ik} g^{jl} - \frac{1}{3} g^{ij} g^{kl} \right) \zeta_{km} \zeta_{ln} g^{mn} \frac{p_\zeta}{\zeta} - \frac{1}{4U_0} \left(\xi^{ij} - \frac{1}{3} g^{ij} \xi \right) \frac{p_\zeta}{\zeta} = \frac{\sqrt{g}}{2\kappa^2} F^{ijkl} \bar{K}_{kl}, \quad (6.10)$$

where we have defined

$$F^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - 8\kappa^2 \alpha U_0 \left(g^{m(i} g^{j)(k} g^{l)n} - \frac{1}{3} g^{ij} g^{km} g^{ln} \right) \zeta_{mo} \zeta_{np} g^{op}. \quad (6.11)$$

F^{ijkl} is symmetric in its first two indices and also in its last two indices:

$$F^{ijkl} = F^{jikl}, \quad F^{ijkl} = F^{ijlk}. \quad (6.12)$$

F^{ijkl} also has the properties

$$g_{ij} F^{ijkl} = g^{kl}, \quad (6.13)$$

$$g_{kl} F^{ijkl} = g^{ij} - 8\kappa^2 \alpha U_0 \left(g^{ik} g^{jl} - \frac{1}{3} g^{ij} g^{kl} \right) \zeta_{km} \zeta_{ln} g^{mn}, \quad (6.14)$$

where the first result is directly related to the fact that both sides of (6.10) are traceless. Finding the inverse F_{ijkl}^{-1} of F^{ijkl} , such that $F_{ijkl}^{-1}F^{klmn} = \delta_{(i}^{(m}\delta_{j)}^{n)} = \delta_i^{(m}\delta_j^{n)}$, enables us to solve \bar{K}_{ij} from (6.10).

In order to prove that the inverse F_{ijkl}^{-1} of (6.11) actually exists, and consequently that no more primary constraints are required, we shall construct F_{ijkl}^{-1} . First note that F_{ijkl}^{-1} must have similar symmetries (6.12) as F^{ijkl} . We can define F_{ijkl}^{-1} as a power series in the coupling $2\kappa^2\alpha U_0$ with the terms depending on the spatial metric g_{ij} and the field ζ_{ij} :

$$F_{ijkl}^{-1} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) + \sum_{J=1}^{\infty} (2\kappa^2\alpha U_0)^J C_{ijkl}^{(J)m_1n_1\dots m_Jn_J} \times \zeta_{m_1o_1}\zeta_{n_1p_1}g^{o_1p_1}\dots\zeta_{m_Jo_J}\zeta_{n_Jp_J}g^{o_Jp_J}, \quad (6.15)$$

where the coefficients $C_{ijkl}^{(J)m_1n_1\dots m_Jn_J}$ depend only on the spatial metric and possess the symmetries of F_{ijkl}^{-1} . These coefficients are defined as follows. The first coefficient is

$$C_{ijkl}^{(1)mn} = 4\left(\delta_{(i}^m g_{j)(k}\delta_l^n) - \frac{1}{3}g_{ij}\delta_{(k}^m\delta_l^n\right) \quad (6.16)$$

and the rest of the coefficients are defined recursively by the formula

$$C_{ijkl}^{(J)m_1n_1\dots m_Jn_J} = C_{ijop}^{(J-1)m_1n_1\dots m_{J-1}n_{J-1}} 4\left(g^{m_J(o}\delta_{(k}^{p)}\delta_l^{n_J)} - \frac{1}{3}g^{op}\delta_{(k}^{m_J}\delta_l^{n_J)}\right) \quad (6.17)$$

for every order $J > 1$. The coefficient $C_{ijkl}^{(J)m_1q_1\dots m_Jq_J}$ of order J in (6.17) vanishes if the coefficient $C_{ijkl}^{(J-1)m_1q_1\dots m_{J-1}q_{J-1}}$ of order $J-1$ is proportional to g_{kl} . In that case the series would be truncated. However, it seems that such a coefficient does not appear and hence the recursion (6.17) will proceed *ad infinitum* for coefficients with increasingly complicated index configurations. For example the second order coefficient is

$$C_{ijkl}^{(2)m_1q_1m_2q_2} = 8\left[\delta_{(i}^{m_1}\delta_{j)}^{m_2}\delta_{(k}^{q_1}\delta_l^{q_2)} + \delta_{(i}^{m_1}g_{j)(k}\delta_l^{q_2)}g^{q_1m_2} - \frac{2}{3}\delta_{(i}^{m_1}\delta_{j)}^{q_1}\delta_{(k}^{m_2}\delta_l^{q_2)} - \frac{1}{3}g_{ij}\left(g^{m_1m_2}\delta_{(k}^{q_1}\delta_l^{q_2)} + g^{q_1m_2}\delta_{(k}^{m_1}\delta_l^{q_2)} - \frac{2}{3}g^{m_1q_1}\delta_{(k}^{m_2}\delta_l^{q_2)}\right)\right]. \quad (6.18)$$

Note that for every order J

$$g^{ij}C_{ijkl}^{(J)m_1n_1\dots m_Jn_J} = 0, \quad (6.19)$$

which implies

$$g^{ij}F_{ijkl}^{-1} = g_{kl}. \quad (6.20)$$

Thus the traceless component of K_{ij} can be written in terms of the canonical variables as

$$\bar{K}_{ij} = \frac{2\kappa^2}{\sqrt{g}}F_{ijkl}^{-1}\bar{P}^{kl}, \quad (6.21)$$

where we introduce

$$\begin{aligned} \bar{P}^{kl} &= p^{ij} - \frac{1}{3}g^{ij}p + \frac{5}{3}\left(g^{ik}g^{jl} - \frac{1}{3}g^{ij}g^{kl}\right)\zeta_{km}\zeta_{ln}g^{mn}\frac{p_\zeta}{\zeta} - \frac{1}{4U_0}\left(\xi^{ij} - \frac{1}{3}g^{ij}\xi\right)\frac{p_\zeta}{\zeta} \\ &= \bar{p}^{kl} + \frac{5}{3}\left(\zeta^k{}_m\zeta^{lm} - \frac{1}{3}g^{kl}\zeta_{mn}\zeta^{mn}\right)\frac{p_\zeta}{\zeta} - \frac{1}{4U_0}\bar{\xi}^{kl}\frac{p_\zeta}{\zeta}, \end{aligned} \quad (6.22)$$

for the purpose of shortening the following expressions for the Hamiltonian and its constraints, which are quite complicated. From now on we may raise and lower spatial indices with the spatial metric, e.g., $\zeta^i_k = g^{ij}\zeta_{jk}$ and $\zeta^{ij} = g^{ik}g^{jl}\zeta_{kl}$, but always keeping in mind that the metric has to be written explicitly when Poisson bracket is evaluated, similarly as it has to be done with the trace components, e.g., $\zeta = g^{ij}\zeta_{ij}$. Together with (6.9), (6.21) and (6.22) express the extrinsic curvature (3.5) in terms of the canonical variables.

Then we shall introduce the Hamiltonian. In the Lagrangian density \mathcal{L} of the action (4.2) the term that contains the time derivative of ζ_{ij} can be written

$$2\alpha U_0 \sqrt{g} g^{ik} g^{jl} \zeta_{kl} K \mathcal{L}_{Nn} \zeta_{ij} = p_\zeta^{ij} \left(\dot{\zeta}_{ij} - \mathcal{L}_N \zeta_{ij} \right), \quad (6.23)$$

where the first term simply cancels $p_\zeta^{ij} \dot{\zeta}_{ij}$ in the Legendre transformation from Lagrangian formalism to Hamiltonian formalism. For the metric we obtain

$$p^{ij} \dot{g}_{ij} = \frac{N}{\sqrt{g}} 4\kappa^2 p^{ij} F_{ijkl}^{-1} \bar{P}^{kl} + \frac{N}{\sqrt{g}} \frac{p}{3\alpha U_0} \frac{p_\zeta}{\zeta} + 2p^{ij} D_{(i} N_{j)}. \quad (6.24)$$

We define the total Hamiltonian as

$$\begin{aligned} H &= \int d^3\mathbf{x} \left(p^{ij} \dot{g}_{ij} + p_\zeta^{ij} \dot{\zeta}_{ij} + \lambda_N p_N + \lambda^i p_i + \lambda_\xi^{ij} p_{ij}^\xi + \bar{v}_{ij} \bar{\Pi}^{ij} - \mathcal{L} \right) \\ &= \int d^3\mathbf{x} \left(N \mathcal{H}_0 + N^i \mathcal{H}_i + \lambda_N p_N + \lambda^i p_i + \lambda_\xi^{ij} p_{ij}^\xi + \bar{v}_{ij} \bar{\Pi}^{ij} \right), \end{aligned} \quad (6.25)$$

where λ_N , λ^i , λ_ξ^{ij} and \bar{v}_{ij} are Lagrange multipliers and we have defined

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{\sqrt{g}} \left[4\kappa^2 p^{ij} F_{ijkl}^{-1} \bar{P}^{kl} + \frac{p}{3\alpha U_0} \frac{p_\zeta}{\zeta} - 2\kappa^2 F_{ijkl}^{-1} \bar{P}^{kl} g^{im} g^{jn} F_{mnop}^{-1} \bar{P}^{op} \right. \\ &\quad + \frac{1}{12\kappa^2 \alpha^2 U_0^2} \left(\frac{p_\zeta}{\zeta} \right)^2 + \frac{4}{9\alpha U_0} \zeta_{ij} \zeta^{ij} \left(\frac{p_\zeta}{\zeta} \right)^2 \\ &\quad + \frac{20\kappa^2}{3} \zeta^{ij} \zeta_j^k F_{iklm}^{-1} \bar{P}^{lm} \frac{p_\zeta}{\zeta} + 16\kappa^4 \alpha U_0 \zeta^{ij} F_{iklm}^{-1} \bar{P}^{lm} \zeta_j^n g^{ko} F_{nopq}^{-1} \bar{P}^{pq} \\ &\quad \left. - \frac{\kappa^2}{U_0} \xi^{ij} F_{ijkl}^{-1} \bar{P}^{kl} \frac{p_\zeta}{\zeta} - \frac{\xi}{12\alpha U_0^2} \left(\frac{p_\zeta}{\zeta} \right)^2 \right] \\ &\quad - \sqrt{g} \left[\frac{R}{2\kappa^2} - 2\alpha U_0 \zeta^{ij} D^k D_k \zeta_{ij} - \alpha \xi^{ij} (\zeta_{ij} - R_{ij}) \right] \end{aligned} \quad (6.26)$$

and

$$\begin{aligned} \mathcal{H}_i &= -2g_{ij} D_k p^{jk} + \partial_i \zeta_{jk} p_\zeta^{jk} - 2\partial_j \left(\zeta_{ik} p_\zeta^{jk} \right) \\ &= -2g_{ij} \partial_k p^{jk} - (2\partial_j g_{ik} - \partial_i g_{jk}) p^{jk} - 2\zeta_{ij} \partial_k p_\zeta^{jk} - (2\partial_j \zeta_{ik} - \partial_i \zeta_{jk}) p_\zeta^{jk}. \end{aligned} \quad (6.27)$$

Observe that the Hamiltonian only depends on the traceless components \bar{p}_ζ^{ij} of the momenta canonically conjugate to ζ_{ij} through the primary constraint (6.8), which also means that \bar{p}_ζ^{ij} does not appear in any other constraint.

The primary constraints (6.1) and (6.8) must be preserved under time evolution generated by the total Hamiltonian (6.25). For this purpose we introduce the secondary

constraints

$$\Phi_0 = \int d^3\mathbf{x} \mathcal{H}_0 \approx 0, \quad (6.28)$$

$$\mathcal{H}_i(\mathbf{x}) \approx 0, \quad (6.29)$$

$$\Psi_{ij}(\mathbf{x}) = \left\{ p_{ij}^\xi(\mathbf{x}), \Phi_0 \right\} \approx 0, \quad (6.30)$$

$$\bar{\Pi}_{IJ}^{ij}(\mathbf{x}) = \left\{ \bar{\Pi}^{ij}(\mathbf{x}), \Phi_0 \right\} \approx 0, \quad (6.31)$$

which ensure that the primary constraints p_N , p_i , p_{ij}^ξ and $\bar{\Pi}^{ij}$ are preserved in time, respectively. The constraint (6.30) can be explicitly defined as the following symmetric tensor density

$$\begin{aligned} \Psi_{ij} = & \frac{1}{\sqrt{g}} \frac{\kappa^2}{U_0} \left(\psi_{ij} - \frac{1}{3} g_{ij} g^{kl} \psi_{kl} + F_{ijkl}^{-1} \bar{P}^{kl} \right) \frac{p_\zeta}{\zeta} \\ & + \frac{1}{\sqrt{g}} \frac{g_{ij}}{12\alpha U_0^2} \left(\frac{p_\zeta}{\zeta} \right)^2 - \sqrt{g} \alpha (\zeta_{ij} - R_{ij}) \approx 0, \end{aligned} \quad (6.32)$$

where we denote

$$\begin{aligned} \psi_{ij} = & p^{kl} F_{kl ij}^{-1} - F_{kl ij}^{-1} g^{km} g^{ln} F_{mnop}^{-1} \bar{P}^{op} + \frac{5}{3} \zeta^{kl} \zeta^m{}_l F_{km ij}^{-1} \frac{p_\zeta}{\zeta} \\ & + 8\kappa^2 \alpha U_0 \zeta^{kl} F_{kl ij}^{-1} \zeta^n{}_l g^{mo} F_{nopq}^{-1} \bar{P}^{pq} - \frac{1}{4U_0} \xi^{kl} F_{kl ij}^{-1} \frac{p_\zeta}{\zeta}. \end{aligned} \quad (6.33)$$

The trace of the constraint Ψ_{ij} is independent of ξ^{ij} . Indeed it can be written as

$$\Psi = \frac{1}{\sqrt{g}} \frac{1}{4\alpha U_0^2} \left(\frac{p_\zeta}{\zeta} \right)^2 - \sqrt{g} \alpha (\zeta - R). \quad (6.34)$$

This implies that

$$\left\{ \Psi(\mathbf{x}), p_{kl}^\xi(\mathbf{y}) \right\} = g^{ij}(\mathbf{x}) \left\{ \Psi_{ij}(\mathbf{x}), p_{kl}^\xi(\mathbf{y}) \right\} = 0. \quad (6.35)$$

The traceless component of the constraint Ψ_{ij} depends on the traceless component of ξ^{ij} , but it is independent of the trace component ξ . Therefore we shall also decompose each of the constraints p_{ij}^ξ and Ψ_{ij} into a traceless component and a trace component:

$$\begin{aligned} p_{ij}^\xi &= \bar{p}_{ij}^\xi + \frac{1}{3} g_{ij} p^\xi, \\ \Psi_{ij} &= \bar{\Psi}_{ij} + \frac{1}{3} g_{ij} \Psi. \end{aligned} \quad (6.36)$$

The explicit form of the constraint $\bar{\Pi}_{IJ}^{ij}$ defined in (6.31) is a very complicated symmetric traceless tensor density. We shall not write it down here since its explicit form will not be used in our analysis. Instead when we evaluate Poisson brackets between $\bar{\Pi}_{IJ}^{ij}$ and the other constraints we shall take advantage of its definition as the Poisson bracket of $\bar{\Pi}^{ij}$ and Φ_0 .

Then we must ensure that every secondary constraint is preserved in time. First we shall consider the constraints \mathcal{H}_i . We introduce a global smeared version of the constraint \mathcal{H}_i as

$$\Phi_S(\chi^i) = \int d^3\mathbf{x} \chi^i \mathcal{H}_i, \quad (6.37)$$

where χ^i ($i = 1, 2, 3$) are arbitrary functions on Σ_t which vanish rapidly enough at infinity. The Poisson brackets of the constraint (6.37) with the canonical variables are

$$\begin{aligned}\{\Phi_S(\chi^k), g_{ij}\} &= -\chi^k \partial_k g_{ij} - \partial_i \chi^k g_{kj} - \partial_j \chi^k g_{ki} = \mathcal{L}_\chi g_{ij}, \\ \{\Phi_S(\chi^k), p^{ij}\} &= -\partial_k \chi^k p^{ij} - \chi^k \partial_k p^{ij} + \partial_k \chi^i p^{kj} + \partial_k \chi^j p^{ki} = \mathcal{L}_\chi p^{ij}, \\ \{\Phi_S(\chi^k), \zeta_{ij}\} &= -\chi^k \partial_k \zeta_{ij} - \partial_i \chi^k \zeta_{kj} - \partial_j \chi^k \zeta_{ki} = \mathcal{L}_\chi \zeta_{ij}, \\ \{\Phi_S(\chi^k), p_\zeta^{ij}\} &= -\partial_k \chi^k p_\zeta^{ij} - \chi^k \partial_k p_\zeta^{ij} + \partial_k \chi^i p_\zeta^{kj} + \partial_k \chi^j p_\zeta^{ki} = \mathcal{L}_\chi p_\zeta^{ij}.\end{aligned}$$

The constraints (6.37) also satisfy the diffeomorphism algebra

$$\{\Phi_S(\chi^i), \Phi_S(\eta^j)\} = \Phi_S(\chi^j \partial_j \eta^i - \eta^j \partial_j \chi^i). \quad (6.38)$$

Thus we identify (6.37) as the momentum constraint that generates diffeomorphisms in the spatial hypersurface Σ_t for the dynamical variables g_{ij} , p^{ij} , ζ_{ij} and p_ζ^{ij} . In fact we can extend the momentum constraint (6.37) to a full generator of spatial diffeomorphisms with the help of the primary constraints (6.1). We redefine

$$\begin{aligned}\Phi_S(\chi^i) &= \int d^3\mathbf{x} \left(p_i \mathcal{L}_\chi N^i + p^{ij} \mathcal{L}_\chi g_{ij} + p_\zeta^{ij} \mathcal{L}_\chi \zeta_{ij} + p_\zeta^{\xi ij} \mathcal{L}_\chi \xi^{ij} \right) \\ &= \int d^3\mathbf{x} \chi^i \Phi_i,\end{aligned} \quad (6.39)$$

where

$$\Phi_i = \mathcal{H}_i + \mathcal{L}_N p_i + 2\partial_j p_{ik}^\xi \xi^{jk} + p_{jk}^\xi \partial_i \xi^{jk} + 2p_{ik}^\xi \partial_j \xi^{jk}. \quad (6.40)$$

Note that the generator for N and p_N vanishes,

$$\int d^3\mathbf{x} p_N \mathcal{L}_\chi N = \int d^3\mathbf{x} \chi^i p_N \partial_i N = 0,$$

since these variables are spatial constants. The extension (6.39) of the momentum constraint simply amounts to rewriting the Lagrange multipliers of the primary constraints (6.1) as

$$\lambda^i = v^i + \mathcal{L}_N N^i, \quad \lambda_\xi^{ij} = v_\xi^{ij} + \mathcal{L}_N \xi^{ij}, \quad \int d^3\mathbf{x} \lambda_N = v_N, \quad (6.41)$$

where v^i , v_ξ^{ij} and v_N are arbitrary. The constraint (6.39) evidently generates time-dependent spatial diffeomorphisms for all the variables. Therefore we obtain

$$\{\Phi_S(\chi^i), \mathcal{H}_0\} = -\chi^i \partial_i \mathcal{H}_0 - \partial_i \chi^i \mathcal{H}_0. \quad (6.42)$$

In other words, \mathcal{H}_0 is a scalar density on the spatial hypersurface Σ_t . We again emphasize that the lapse N depends only on time. As a result there is no local Hamiltonian constraint \mathcal{H}_0 but only the global one Φ_0 defined in (6.28). As a result we clearly have

$$\{\Phi_0, \Phi_0\} = 0, \quad (6.43)$$

$$\{\Phi_S(\chi^i), \Phi_0\} = 0. \quad (6.44)$$

It is now evident that the momentum constraint (6.40) is preserved in time, since (6.43) and (6.44) and all the rest of the constraints in the Hamiltonian transform as scalar or

tensor densities under spatial diffeomorphism. Indeed we can write the Hamiltonian as a sum of the constraints

$$H = N\Phi_0 + \Phi_S(N^i) + v_N p_N + \int d^3\mathbf{x} \left(v^i p_i + v_\xi^{ij} p_{ij}^\xi + \bar{v}_{ij} \bar{\Pi}^{ij} \right). \quad (6.45)$$

We could also include the 11 secondary constraints Ψ_{ij} and $\bar{\Pi}_{II}^{ij}$ into the Hamiltonian with arbitrary Lagrange multipliers. However, those 11 Lagrange multipliers would already be fixed by the 11 consistency conditions for the primary constraints p_{ij}^ξ and $\bar{\Pi}^{ij}$. Thus the inclusion of those constraints is of no benefit to us. Hence we leave them out from the Hamiltonian.

The global Hamiltonian constraint Φ_0 is preserved in time due to the constraints p_{ij}^ξ , Ψ_{ij} , $\bar{\Pi}^{ij}$, $\bar{\Pi}_{II}^{ij}$.

$$\{\Phi_0, H\} \approx - \int d^3\mathbf{x} \left(v_\xi^{ij} \Psi_{ij} + \bar{v}_{ij} \bar{\Pi}_{II}^{ij} \right) \approx 0. \quad (6.46)$$

Next we have to ensure that the secondary constraint Ψ_{ij} is preserved in time. The Poisson brackets $\{\bar{\Psi}_{ij}, H\}$ and $\{\Psi, H\}$ must both be either zero or a constraint. The consistency conditions can be written as

$$N \{ \bar{\Psi}_{ij}(\mathbf{x}), \Phi_0 \} + \int d^3\mathbf{y} \left(v_\xi^{kl}(\mathbf{y}) \{ \bar{\Psi}_{ij}(\mathbf{x}), p_{kl}^\xi(\mathbf{y}) \} + \bar{v}_{kl}(\mathbf{y}) \{ \bar{\Psi}_{ij}(\mathbf{x}), \bar{\Pi}^{kl}(\mathbf{y}) \} \right) = 0 \quad (6.47)$$

and

$$N \{ \Psi(\mathbf{x}), \Phi_0 \} + 3\alpha\sqrt{g} \frac{\bar{v}_{ij} \bar{\zeta}^{ij}}{\zeta}(\mathbf{x}) = 0 \quad (6.48)$$

for the traceless component $\bar{\Psi}_{ij}$ and the trace component Ψ , respectively. In (6.48), we used (6.35) and

$$\{ \Psi(\mathbf{x}), \bar{\Pi}^{ij}(\mathbf{y}) \} = 3\alpha\sqrt{g} \frac{\bar{\zeta}^{ij}}{\zeta}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}). \quad (6.49)$$

The consistency condition (6.48) can be solved for one of the 5 independent components in the Lagrange multiplier \bar{v}_{ij} . Let us denote the specific solution to (6.48) by \bar{v}'_{ij} , which still contains 4 independent arbitrary components. The solved component of \bar{v}'_{ij} is proportional to N . The homogeneous part of (6.48) has only the nontrivial solution. Consistency of Ψ is assured by substituting the solution $\bar{v}_{ij} = \bar{v}'_{ij}$ into the Hamiltonian. Then the consistency of the constraint $\bar{\Psi}_{ij}$ is assured by solving the inhomogeneous linear equation (6.47) for the Lagrange multiplier v_ξ^{kl} . None of the three Poisson brackets in (6.47) vanishes, not even weakly. Explicitly the second Poisson bracket can be written as

$$\begin{aligned} \{ \bar{\Psi}_{ij}(\mathbf{x}), p_{kl}^\xi(\mathbf{y}) \} = & -\frac{1}{\sqrt{g}} \frac{\kappa^2}{4U_0^2} \left(A_{ijkl} - \frac{1}{3} g_{ij} g^{mn} A_{mnkl} \right. \\ & \left. + F_{ijkl}^{-1} - \frac{1}{3} g_{kl} F_{ijmn}^{-1} g^{mn} \right) \left(\frac{p_\zeta}{\zeta} \right)^2(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (6.50)$$

where we denote

$$\begin{aligned} A_{ijkl} = & F_{klij}^{-1} - F_{mnij}^{-1} g^{mo} g^{np} \left(F_{opkl}^{-1} - \frac{1}{3} g_{kl} F_{opqr}^{-1} g^{qr} \right) \\ & + 8\kappa^2 \alpha U_0 \zeta^{mn} F_{m o i j}^{-1} \zeta^p_n g^{oq} \left(F_{pqkl}^{-1} - \frac{1}{3} g_{kl} F_{pqrs}^{-1} g^{rs} \right). \end{aligned} \quad (6.51)$$

Thus we obtain

$$g^{kl}(\mathbf{y}) \left\{ \bar{\Psi}_{ij}(\mathbf{x}), p_{kl}^\xi(\mathbf{y}) \right\} = 0, \quad (6.52)$$

where we also used the property (6.20). Clearly the homogeneous part of the equation (6.47)

$$\int d^3\mathbf{y} v_\xi^{kl}(\mathbf{y}) \left\{ \bar{\Psi}_{ij}(\mathbf{x}), p_{kl}^\xi(\mathbf{y}) \right\} = 0 \quad (6.53)$$

has the solution

$$v_\xi^{kl} = \frac{1}{3} v_\xi g^{kl}, \quad (6.54)$$

where the trace component v_ξ of the Lagrange multiplier v_ξ^{kl} is still an arbitrary field. This is the only nontrivial solution to the homogeneous equation (6.53). The specific solution to the inhomogeneous equation (6.47) has the form

$$v_\xi^{kl} = N \bar{w}_\xi^{kl} [g_{ij}, p^{ij}, \zeta_{ij}, p_\zeta, \xi^{ij}, \bar{v}'_{ij}/N], \quad (6.55)$$

where \bar{w}_ξ^{kl} is a traceless functional of the listed variables and of the Lagrange multiplier \bar{v}'_{ij} divided by N . Formally it can be written as

$$\begin{aligned} \bar{w}_\xi^{ij}(\mathbf{x}) = & - \int d^3\mathbf{y} B^{ijkl}(\mathbf{x}, \mathbf{y}) \left\{ \bar{\Psi}_{kl}(\mathbf{y}), \Phi_0 \right\} \\ & - \iint d^3\mathbf{y} d^3\mathbf{z} \frac{\bar{v}'_{kl}(\mathbf{z})}{N} B^{ijmn}(\mathbf{x}, \mathbf{y}) \left\{ \bar{\Psi}_{mn}(\mathbf{y}), \bar{\Pi}^{kl}(\mathbf{z}) \right\}, \end{aligned} \quad (6.56)$$

where $B^{ijkl}(\mathbf{x}, \mathbf{y})$ is the inverse to (6.50), i.e., it satisfies

$$\int d^3\mathbf{y} B^{ijkl}(\mathbf{x}, \mathbf{y}) \left\{ \bar{\Psi}_{kl}(\mathbf{y}), p_{mn}^\xi(\mathbf{z}) \right\} = \delta_m^{(i} \delta_n^{j)} \delta(\mathbf{x} - \mathbf{z}). \quad (6.57)$$

The general solution to (6.47) is the sum of the specific solution (6.55) and the solution (6.54) to the homogeneous equation (6.53):

$$v_\xi^{kl} = N \bar{w}_\xi^{kl} [g_{ij}, p^{ij}, \zeta_{ij}, p_\zeta, \xi^{ij}, \bar{v}'_{ij}/N] + \frac{1}{3} v_\xi g^{kl}. \quad (6.58)$$

Inserting this solution into the Hamiltonian (6.45) ensures the consistency of $\bar{\Psi}_{ij}$. The Hamiltonian is now written as

$$H = N\Phi_0 + N \int d^3\mathbf{x} \bar{w}_\xi^{ij} \bar{p}_{ij}^\xi + \Phi_S(N^i) + v_N p_N + \int d^3\mathbf{x} \left(v^i p_i + \frac{1}{3} v_\xi p^\xi + \bar{v}'_{ij} \bar{\Pi}^{ij} \right). \quad (6.59)$$

Lagrange multipliers that are arbitrary or at least contain some arbitrary components are denoted by v and the multipliers denoted by w have been solved entirely in order to ensure the consistency of constraints, e.g., in (6.59) \bar{w}_ξ^{ij} is given by the specific solution (6.56) to (6.47) and \bar{v}'_{ij} is the solution to (6.48) with four arbitrary components left.

As a last step in Dirac's algorithm we have to ensure the consistency of the secondary constraints (6.31) in time. We require that

$$\begin{aligned} \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), H \right\} \approx & N \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), \Phi_0 \right\} + N \int d^3\mathbf{y} \bar{w}_\xi^{kl}(\mathbf{y}) \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), \bar{p}_{kl}^\xi(\mathbf{y}) \right\} \\ & + \int d^3\mathbf{y} \frac{1}{3} v_\xi(\mathbf{y}) \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), p^\xi(\mathbf{y}) \right\} + \int d^3\mathbf{y} \bar{v}'_{kl}(\mathbf{y}) \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), \bar{\Pi}^{kl}(\mathbf{y}) \right\} \end{aligned} \quad (6.60)$$

must be either zero or a constraint. In this equation \bar{w}_{ξ}^{kl} is given by (6.56). For the calculation of the Poisson brackets of $\bar{\Pi}_{II}^{ij}$ with p^{ξ} and \bar{p}_{kl}^{ξ} we can use the definition of $\bar{\Pi}_{II}^{ij}(\mathbf{x})$ as the Poisson bracket $\{\bar{\Pi}^{ij}(\mathbf{x}), \Phi_0\}$, and the Jacobi identity for the Poisson bracket. First we obtain

$$\begin{aligned} \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), p_{kl}^{\xi}(\mathbf{y}) \right\} &= - \left\{ \left\{ \Phi_0, p_{kl}^{\xi}(\mathbf{y}) \right\}, \bar{\Pi}^{ij}(\mathbf{x}) \right\} - \left\{ \left\{ p_{kl}^{\xi}(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\}, \Phi_0 \right\} \\ &= \left\{ \Psi_{kl}(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\} \\ &= \left\{ \bar{\Psi}_{kl}(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\} + \frac{1}{3} g_{kl}(\mathbf{y}) \left\{ \Psi(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\}, \end{aligned} \quad (6.61)$$

where we also used the independence of $\bar{\Pi}^{ij}$ on ξ^{ij} . For the trace component p^{ξ} we obtain

$$\begin{aligned} \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), p^{\xi}(\mathbf{y}) \right\} &= - \left\{ \left\{ \Phi_0, p^{\xi}(\mathbf{y}) \right\}, \bar{\Pi}^{ij}(\mathbf{x}) \right\} - \left\{ \left\{ p^{\xi}(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\}, \Phi_0 \right\} \\ &= - \left\{ \left\{ \Phi_0, g^{kl}(\mathbf{y}) \right\}, \bar{\Pi}^{ij}(\mathbf{x}) \right\} p_{kl}^{\xi}(\mathbf{y}) + \left\{ \Psi(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\} \\ &\approx \left\{ \Psi(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\}, \end{aligned} \quad (6.62)$$

where the last expression is given in (6.49). For the traceless component \bar{p}_{ij}^{ξ} we obtain

$$\begin{aligned} \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), \bar{p}_{kl}^{\xi}(\mathbf{y}) \right\} &= \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), p_{kl}^{\xi}(\mathbf{y}) \right\} - \frac{1}{3} \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), g_{kl}(\mathbf{y}) \right\} p^{\xi}(\mathbf{y}) \\ &\quad - \frac{1}{3} g_{kl}(\mathbf{y}) \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), p^{\xi}(\mathbf{y}) \right\} \\ &\approx \left\{ \bar{\Psi}_{kl}(\mathbf{y}), \bar{\Pi}^{ij}(\mathbf{x}) \right\}, \end{aligned} \quad (6.63)$$

which is a complicated nonvanishing expression. The first and the last Poisson brackets in (6.60) are neither zero nor a combination of the constraints. Indeed they are very complicated expressions. Thus we see that the 5 consistency conditions for $\bar{\Pi}_{II}^{ij}$ should be solved for the Lagrange multiplier v_{ξ} and the 4 arbitrary Lagrange multipliers left in \bar{v}'_{kl} . We obtain the following equations for them

$$\begin{aligned} N \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), \Phi_0 \right\} + N \int \int d^3 \mathbf{y} d^3 \mathbf{z} \left\{ \bar{\Pi}^{ij}(\mathbf{x}), \bar{\Psi}_{kl}(\mathbf{y}) \right\} B^{klmn}(\mathbf{y}, \mathbf{z}) \left\{ \bar{\Psi}_{mn}(\mathbf{z}), \Phi_0 \right\} \\ + \int \int \int d^3 \mathbf{y} d^3 \mathbf{z} d^3 \mathbf{z}' \bar{v}_{kl}(\mathbf{z}') \left\{ \bar{\Pi}^{ij}(\mathbf{x}), \bar{\Psi}_{mn}(\mathbf{y}) \right\} B^{mnop}(\mathbf{y}, \mathbf{z}) \left\{ \bar{\Psi}_{op}(\mathbf{z}), \bar{\Pi}^{kl}(\mathbf{z}') \right\} \\ + \alpha \sqrt{g} \frac{\bar{\zeta}^{ij}}{\zeta} v_{\xi}(\mathbf{x}) + \int d^3 \mathbf{y} \bar{v}'_{kl}(\mathbf{y}) \left\{ \bar{\Pi}_{II}^{ij}(\mathbf{x}), \bar{\Pi}^{kl}(\mathbf{y}) \right\} = 0. \end{aligned} \quad (6.64)$$

The general solution to (6.64) has the form

$$\begin{aligned} v_{\xi} &= N w \left[g_{ij}, p^{ij}, \zeta_{ij}, p_{\zeta}, \xi^{ij} \right], \\ \bar{v}'_{kl} &= N \bar{w}_{kl} \left[g_{ij}, p^{ij}, \zeta_{ij}, p_{\zeta}, \xi^{ij} \right] + f \bar{h}_{kl} \left[g_{ij}, p^{ij}, \zeta_{ij}, p_{\zeta}, \xi^{ij} \right], \end{aligned} \quad (6.65)$$

where w and \bar{w}_{kl} comprise the specific solution to (6.64) (divided by N), \bar{h}_{kl} is a possible nontrivial solutions to the part of (6.64) that is homogeneous in \bar{v}'_{kl} , and f is an arbitrary functions of time. Because of the very complicated form of the equations (6.64), we have not been able to rigorously establish the existence of neither the specific solution nor the homogeneous solutions. Still we assume that a specific solution exists. It is unclear whether nontrivial solution \bar{h}_{kl} to the homogeneous part of (6.64) exists. Conceivably there could even be several solutions to the homogeneous part. The existence

of a solution to the homogeneous equation, would imply the existence of an extra gauge symmetry associated with a global (integrated) first-class constraint, a generator of the gauge transformation. We, however, suspect that such a gauge symmetry does not exist in the theory. Nevertheless if homogeneous solutions do exist they can be dealt with by introducing some global gauge fixing conditions. Hence we shall continue our analysis as if no nontrivial solution to the part of (6.64) that is homogeneous in \bar{v}'_{kl} exists, i.e., we assume $\bar{h}_{kl} = 0$.

The Hamiltonian that ensures the consistency of all the constraints in time is

$$H = N\Phi_0 + N \int d^3\mathbf{x} \left(\bar{w}_\xi^{ij} \bar{p}_{ij}^\xi + \frac{1}{3} w p^\xi + \bar{w}_{ij} \bar{\Pi}^{ij} \right) + \Phi_S(N^i) + v_N p_N + \int d^3\mathbf{x} v^i p_i, \quad (6.66)$$

where $N\bar{w}_\xi^{ij}$ is the (traceless) specific solution (6.56) to the inhomogeneous equation (6.47) and Nw and $N\bar{w}_{ij}$ are the specific solution to the inhomogeneous equations (6.48) and (6.64). Since Hamiltonian is always a first-class quantity and (6.66) is a sum of constraints, we can conclude that the Hamiltonian is a sum of first-class constraints. That is exactly what one expects from a system with time reparameterization invariance. We already knew that the momentum constraint \mathcal{H}_i , or rather its generalization Φ_i in (6.40), and the primary constraints p_N and p_i are first-class constraints. Now we can see that the linear combination

$$\Phi_0 + \int d^3\mathbf{x} \left(\bar{w}_\xi^{ij} \bar{p}_{ij}^\xi + \frac{1}{3} w p^\xi + \bar{w}_{ij} \bar{\Pi}^{ij} \right) = \int d^3\mathbf{x} \left(\mathcal{H}_0 + \bar{w}_\xi^{ij} \bar{p}_{ij}^\xi + \frac{1}{3} w p^\xi + \bar{w}_{ij} \bar{\Pi}^{ij} \right) \quad (6.67)$$

is a first-class constraint. All these first-class constraints are associated with the symmetry under foliation preserving diffeomorphism: invariance under time-dependent spatial diffeomorphism and time reparameterization. In addition a global first-class constraint associated with each solution to the homogeneous part of the equation (6.64) could conceivably exist, although we suspected that such solutions do not exist.

The nature of the rest of the secondary constraints remains to be elaborated. Namely we are interested in whether some linear combination of the secondary constraints is a first-class constraint. That is in addition to the known first-class combination (6.67), which is indeed the only one that can include Φ_0 . Under the Hamiltonian (6.45) we remarked that including Ψ_{ij} and $\bar{\Pi}_{IJ}^{ij}$ in the Hamiltonian with Lagrange multipliers would result to the fixation of the Lagrange multipliers due to the consistency conditions of the primary constraint p_{ij}^ξ and $\bar{\Phi}^{ij}$. This shows that every linear combination of these constraints is a second-class constraint. Consequently there is no need to extend the Hamiltonian (6.66) with further secondary constraints.

Finally we can seek to identify the physical degrees of freedom. First let us count the number of propagating physical degrees of freedom (physical d.o.f.) by using Dirac's formula:

$$\begin{aligned} \#(\text{physical d.o.f.}) = \frac{1}{2} [\#(\text{canonical variables}) - 2 \times \#(\text{first-class constraints}) \\ - \#(\text{second-class constraints})] . \end{aligned} \quad (6.68)$$

We have 42 \mathbf{x} -dependent canonical variables (N^i , g_{ij} , ζ_{ij} , ξ^{ij} and their conjugated momenta), 6 first-class constraints (p_i , Φ_i) and 22 second-class constraints (p_{ij}^ξ , $\bar{\Pi}^{ij}$, Ψ_{ij} , $\bar{\Pi}_{IJ}^{ij}$):

$$\#(\text{physical d.o.f.}) = \frac{42 - 12 - 22}{2} = 4. \quad (6.69)$$

That is 2 more physical degrees of freedom than in GR. For \mathbf{x} -independent zero modes there exist two more variables N, p_N and two more first-class constraints p_N and (6.67), thus yielding 3 physical degrees of freedom, one more than in GR. Let us further seek to understand the nature of the extra modes in the theory.

The constraints of the theory enable us to regard some of the canonical variables as being dependent on other variables. By analysing the dependencies of the variables we seek to identify the physical degrees of freedom. First let us consider the second-class constraints. We assume that any conceivable first-class constraint associated with a solution to the homogeneous part of (6.64) has been dealt with by introducing a global gauge fixing condition, so that no first-class constraint is set to zero strongly when we impose the second-class constraints to zero locally, and hence the Dirac bracket will be well defined. The Dirac bracket can be defined in the standard way. Then we can set the local second-class constraints p_{ij}^ξ , $\bar{\Pi}^{ij}$, Ψ_{ij} and $\bar{\Pi}_{II}^{ij}$ to zero. Let us list the variables that are regarded as dependent variables:

- The momenta p_{ij}^ξ canonically conjugate to ξ^{ij} can be set to zero. However, they did not appear in the Hamiltonian in the first place.
- The traceless component of ξ^{ij} can be solved from $\bar{\Psi}_{ij}$.
- Ψ can be regarded to constrain the metric g_{ij} . For instance we can regard that Ψ fixes the conformal factor of the metric g_{ij} . Note that we cannot use Ψ to solve ξ because Ψ is independent of ξ^{ij} .
- The constraints $\bar{\Pi}^{ij}$ and $\bar{\Pi}_{II}^{ij}$ can be regarded to define the traceless variables $\bar{\zeta}_{ij}$ and \bar{p}_ζ^{ij} in terms of independent variables.

Now the remaining independent canonical variables are 11 variables in g_{ij} and p^{ij} , the trace component ξ , and the trace components ζ and p^ζ . Surprisingly we could not remove ξ from the set of independent variables even though the variables ξ^{ij} have a purely auxiliary role in the action (4.2).

Recall that in GR the canonical variables g_{ij} and p^{ij} are restricted by the Hamiltonian constraint and the momentum constraint so that we can freely specify Cauchy data for the trace and transverse-traceless parts of the momentum p^{ij} and the also spatial metric g_{ij} up to a conformal factor [40–42]. Then the longitudinal (vector) part of the momentum p^{ij} and the conformal factor of metric are fixed by the momentum constraint and the Hamiltonian constraint, respectively. Fixing the trace p of the momentum fixes the slicing of space-time into spacelike hypersurfaces [40, 43, 44]; the scalar quantity $g^{-1/2}p$ measures the rate of contraction of local three-volume elements with respect to local proper time. This leaves us the two well-known gravitational degrees of freedom which can be described by a pair of symmetric transverse traceless tensors.

The remaining degeneracy left in the system is due to the time-dependent spatial diffeomorphism generated by the momentum constraint and the invariance under time reparameterization. Imposing the momentum constraint removes 3 degrees of freedom. Fixing the time reparameterization symmetry does not affect local dynamics. Thus the spatial metric tensor together with ξ carries 3 physical degrees of freedom. The fact that the local Hamiltonian constraint is absent in the given theory implies that there should exist an additional scalar mode as in projectable version of HL gravity. But recall that here the projectability of N appeared because we chose to work with a foliation adapted to the field ϕ so that the spatial hypersurfaces are those of constant ϕ . By solving the

equation of motion (2.3) we have effectively chosen the direction of the time coordinate. Using an arbitrary spacelike foliation would not imply the projectability of N . Then of course there would appear higher derivatives of ϕ in the action. In that case the extra mode would presumably be associated with ϕ itself. In addition there exists another extra degree of freedom in the trace component ζ of the tensor field ζ_{ij} . This extra mode is clearly associated with the second order time derivatives present in the action (3.29).

In the gauge $N = 1, N^i = 0$, the Hamiltonian in reduced phase space is

$$H = \Phi_0 = \int d^3\mathbf{x} \mathcal{H}_0, \quad (6.70)$$

where \mathcal{H}_0 is expressed in terms of the independent variables in g_{ij} , p^{ij} , ξ and ζ , p_ζ . However, we shall soon see that the Hamiltonian can be written without dependence on ξ , due to the constraints.

The fact that a given higher derivative theory of gravity possesses extra degrees of freedom is always alarming, because such modes are often ghosts or otherwise pathological. Therefore it should be checked whether the extra modes present in this new version of CRG are ghosts, particularly the propagating mode ζ of the higher-order variable. We can see that the kinetic part of the Hamiltonian contains terms involving the momentum p_ζ which can attain arbitrarily negative values on the constraint surface. Note that this is not quite as evident as it would seem at first sight because of the existence of complicated constraints. Indeed a better view of this point is achieved by taking out a linear combination of the constraints Ψ_{ij} from the Hamiltonian:

$$\begin{aligned} \mathcal{H}_0 + \xi^{ij}\Psi_{ij} = & \frac{1}{\sqrt{g}} \left[4\kappa^2 p^{ij} F_{ijkl}^{-1} \bar{\mathcal{P}}^{kl} + \frac{p}{3\alpha U_0} \frac{p_\zeta}{\zeta} - 2\kappa^2 F_{ijkl}^{-1} \bar{\mathcal{P}}^{kl} g^{im} g^{jn} F_{mnop}^{-1} \bar{\mathcal{P}}^{op} \right. \\ & + \frac{\kappa^2}{8U_0^2} F_{ijkl}^{-1} \bar{\zeta}^{kl} g^{im} g^{jn} F_{mnop}^{-1} \bar{\zeta}^{op} \left(\frac{p_\zeta}{\zeta} \right)^2 + \frac{1}{12\kappa^2 \alpha^2 U_0^2} \left(\frac{p_\zeta}{\zeta} \right)^2 \\ & + \frac{4}{9\alpha U_0} \zeta_{ij} \zeta^{ij} \left(\frac{p_\zeta}{\zeta} \right)^2 + \frac{20\kappa^2}{3} \zeta^{ij} \zeta^k{}_j F_{iklm}^{-1} \bar{\mathcal{P}}^{lm} \frac{p_\zeta}{\zeta} \\ & + 16\kappa^4 \alpha U_0 \zeta^{ij} F_{iklm}^{-1} \bar{\mathcal{P}}^{lm} \zeta^n{}_j g^{ko} F_{nopq}^{-1} \bar{\mathcal{P}}^{pq} \\ & \left. - \frac{\kappa^4 \alpha}{U_0} F_{iklm}^{-1} \bar{\zeta}^{lm} \zeta^n{}_j g^{ko} F_{nopq}^{-1} \bar{\zeta}^{pq} \left(\frac{p_\zeta}{\zeta} \right)^2 - \frac{\kappa^2}{U_0} \bar{\zeta}^{ij} F_{ijkl}^{-1} \bar{\zeta}^{kl} \left(\frac{p_\zeta}{\zeta} \right)^2 \right] \\ & - \sqrt{g} \left(\frac{R}{2\kappa^2} - 2\alpha U_0 \zeta^{ij} D^k D_k \zeta_{ij} \right) \end{aligned} \quad (6.71)$$

where we denote

$$\bar{\mathcal{P}}^{kl} = \bar{p}^{kl} + \frac{5}{3} \left(\zeta^k{}_m \zeta^{lm} - \frac{1}{3} g^{kl} \zeta_{mn} \zeta^{mn} \right) \frac{p_\zeta}{\zeta}. \quad (6.72)$$

This suggests that there exists a degree of freedom that carries negative energy. As is usual in higher time derivative theories, energy of the higher-order degree of freedom has opposite sign compared to the corresponding lower order degree of freedom. Here these gravitational degrees of freedom evidently interact with each other. We emphasize that the Hamiltonian is not the energy of the gravitational system, but rather a constraint that vanishes everywhere on the constraint surface. The ADM energy of an asymptotically flat gravitational system can be obtained by studying the surface terms of the Hamiltonian with appropriate boundary conditions. The problem is not that the Hamiltonian or the

total energy would attain arbitrarily negative values. The real source of the problem is that any state can decay into compensating positive and negative energy excitations. Even “empty space” will decay into a tempest of positive and negative energy excitations. This makes the theory unstable. The only way this could be avoided are the constraints. Unfortunately the Hamiltonian constraint is global and therefore it does not affect local physics. We suspect the momentum constraints does not protect the stability either.

We note that the Hamiltonian formulation of CRG possesses a singularity at $\zeta_{ij} = 0$. In the subspace $\zeta_{ij} = 0$ of phase space, all the momenta p_ζ^{ij} are primary constraints and therefore there is no degree of freedom. Thus there is a kind of discontinuity between the solutions discussed in Sec. 5 and the ones with nonvanishing ζ_{ij} .

Some characteristics of the CRG action (4.2) can be demonstrated by the following simple Lagrangian with two degrees of freedom $q^a(t)$ ($a = 1, 2$)

$$L(q^a, \dot{q}^a) = \frac{1}{2} A_{ab} \dot{q}^a \dot{q}^b - V(q^a), \quad A = \begin{pmatrix} 1 & \alpha q^2 \\ \alpha q^2 & 0 \end{pmatrix}, \quad (6.73)$$

where α is a nonvanishing constant. The canonically conjugated momenta are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = A_{ab} \dot{q}^b. \quad (6.74)$$

When $q^2 \neq 0$, A has an inverse and we obtain the Hamiltonian

$$H(q^a, p_a) = \frac{p_1 p_2}{\alpha q^2} - \frac{1}{2} \left(\frac{p_2}{\alpha q^2} \right)^2 + V(q^a). \quad (6.75)$$

In the limit $q^2 \rightarrow 0$, the kinetic part of the Hamiltonian diverges. Indeed when $q^2 = 0$, the conjugated momentum $p_2 = 0$ and the model reduces to a single degree of freedom q^1 . This is analogous to the case $\zeta_{ij} = 0$ of CRG. When $q^2 \neq 0$, equations of motion are

$$\begin{aligned} \dot{q}^1 &= \frac{p_2}{\alpha q^2}, & \dot{p}_1 &= -\frac{\partial V}{\partial q_1}, \\ \dot{q}^2 &= \frac{p_1}{\alpha q^2} - \frac{p_2}{(\alpha q^2)^2}, & \dot{p}_2 &= \frac{p_1 p_2}{\alpha (q^2)^2} - \frac{p_2^2}{\alpha^2 (q^2)^3} - \frac{\partial V}{\partial q_2}. \end{aligned} \quad (6.76)$$

The Hamiltonian is not bounded below even when we consider solutions where q^2 retains its sign under time evolution. The kinetic matrix A has both a positive and a negative eigenvalue

$$\lambda_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 + 4(\alpha q^2)^2} \right). \quad (6.77)$$

Thus we can write the Lagrangian as

$$L = \frac{\lambda_+}{2} (\dot{Q}^+)^2 + \frac{\lambda_-}{2} (\dot{Q}^-)^2 - \tilde{V}(Q^\pm) \quad (6.78)$$

where the variables Q^\pm are defined by

$$\dot{Q}^\pm = u_a^\pm \dot{q}^a, \quad u^\pm = \left(\frac{\lambda_\pm}{(\alpha q^2)^2} + 2 \right)^{-\frac{1}{2}} \left(\frac{\lambda_\pm}{\alpha q^2}, 1 \right). \quad (6.79)$$

Thus the system has a positive energy and a negative energy degree of freedom. If this model were a continuum field theory it would certainly be unstable, assuming the two modes interact with each other.

We expect the CRG actions corresponding to higher values of the critical exponent $z > 3$ to exhibit similar problems as the case $z = 3$. As was already noted in Sec. 3, the number of time derivatives present in the ADM representation of the action grows with z . For example for $z = 4, 5, 6$ we are most likely to recover the same problems but in an even more complicated form than in the case $z = 3$. CRG actions corresponding to sufficiently high z will necessarily be unstable, once the order of time derivatives is greater than the number of available constraints.

In more general perspective, we conjecture that generally covariant higher derivative theories of gravity which involve spontaneous (constraint induced) Lorentz and/or diffeomorphism symmetry breaking will in general share the ghost problem with CRG. The only way this could be avoided is that either the higher time derivatives totally disappear (cancel out) from the ADM representation of the given action or the constraints available after symmetry breaking conspire to protect the stability of the higher-order degrees of freedom. The former way out is unlikely to be realized in $d = 4$ and higher spacetime dimensions because at least $2(d - 1)$ derivatives are required in the invariants present in the generally covariant action of the theory in order to achieve multiplicative renormalizability. The latter way out is also hard to achieve especially when diffeomorphism invariance is broken and hence the constraint content is no longer similar to GR. Still it is not entirely impossible that such a theory could be conceived when our understanding on this kind of theories grows.

7 Alternative variables for Hamiltonian formulation

Instead of (4.1) alternative variables could be used to absorb the second order time derivatives in the action (3.29) in order to construct Hamiltonian formulation of new CRG. Here we briefly consider two more choices of higher-order variables.

7.1 Scalar variable

Since the second order time derivatives in the action (3.29) can be written

$$\begin{aligned} 2(R^{ij} + KK^{ij}) \partial_t (R_{ij} + KK_{ij}) &= \partial_t [(R^{ij} + KK^{ij}) (R_{ij} + KK_{ij})] \\ &+ 4N (R^{ij} + KK^{ij}) K_{ik} g^{kl} (R_{lj} + KK_{lj}) \\ &+ 4 (R^{ij} + KK^{ij}) D_{(i} N_{k)} g^{kl} (R_{lj} + KK_{lj}) \end{aligned} \quad (7.1)$$

the following scalar variable can be used to absorb the second order time derivatives

$$\zeta = (R^{ij} + KK^{ij}) (R_{ij} + KK_{ij}) . \quad (7.2)$$

Now the action can be replaced by

$$\begin{aligned} S_3 = \int d^4x \sqrt{g} N \left\{ \frac{K_{ij} K^{ij} - K^2 + R}{2\kappa^2} + \alpha U_0 \left[\frac{K}{N} (\dot{\zeta} - \mathcal{L}_N \zeta) - D^i D_i \zeta \right. \right. \\ \left. \left. - 4 (R^{ij} + KK^{ij}) K_{ik} K^{kl} (R_{lj} + KK_{lj}) \right] \right. \\ \left. + \alpha \xi [\zeta - (R^{ij} + KK^{ij}) (R_{ij} + KK_{ij})] \right\} , \quad (7.3) \end{aligned}$$

where the scalar field ξ is a Lagrange multiplier. This action has the disadvantage that it involves the extrinsic curvature to the sixth power. As a result the momenta canonically conjugate to the spatial metric will involve extrinsic curvature to the fifth power. Obtaining the Hamiltonian via Legendre transformation thus becomes a considerable problem. On the other hand the number of second-class constraints would be much lower than in the formalism of Sec. 6.

However, the given form of the action is suitable for illustrating the question whether the theory contains ghosts or not. Let us consider the following expression that contains the only time derivative of the second order variable ζ in the action

$$K\dot{\zeta} = \frac{1}{2} \begin{pmatrix} K & \dot{\zeta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} K \\ \dot{\zeta} \end{pmatrix} \quad (7.4)$$

Introducing two scalar fields ϕ_1 and ϕ_2 as

$$\dot{\phi}_1 = \frac{1}{\sqrt{2}}(K - \dot{\zeta}), \quad \dot{\phi}_2 = \frac{1}{\sqrt{2}}(K + \dot{\zeta}), \quad (7.5)$$

enables us to diagonalize of this kinetic term. We obtain

$$\alpha U_0 K \dot{\zeta} = \frac{\alpha U_0}{2} (\dot{\phi}_2^2 - \dot{\phi}_1^2). \quad (7.6)$$

Depending on the sign of the coupling constant α it seems that either ϕ_1 or ϕ_2 has a negative kinetic term which is an indication that given theory contains a ghost mode. But this is just the diagonalization of the cross-term between K and $\dot{\zeta}$. We also have to substitute

$$K = \frac{1}{\sqrt{2}} (\dot{\phi}_1 + \dot{\phi}_2), \quad K_{ij} = \bar{K}_{ij} + \frac{g_{ij}}{3\sqrt{2}} (\dot{\phi}_1 + \dot{\phi}_2) \quad (7.7)$$

into the whole kinetic part of the action (7.3). One obtains that the action contains $\dot{\phi}_1$ and $\dot{\phi}_2$ up to sixth power, including many cross-terms between $\dot{\phi}_1$ and $\dot{\phi}_2$. The sign of the kinetic terms with $\dot{\phi}_1$ and $\dot{\phi}_2$ to powers higher than second order is $-\text{sgn}(\alpha)$. In particular the terms that contains the highest powers are contained in the term

$$-\frac{\alpha U_0}{54} (\dot{\phi}_1 + \dot{\phi}_2)^6. \quad (7.8)$$

Thus when $\alpha > 0$ the kinetic part does not possesses a stable minimum, while for $\alpha < 0$ it does. However, answering the question of ghosts may be more involved due to the presence of velocities to powers higher than two. Therefore the quadratic action (4.2) remains a safer alternative.

7.2 Extrinsic curvature as an additional variable

Yet another possible choice is to use the extrinsic curvature K_{ij} as an additional variable

$$\zeta_{ij} = K_{ij}. \quad (7.9)$$

Then all the time derivatives in the action (3.29) will be absorbed into ζ_{ij} and $\dot{\zeta}_{ij}$. The time derivative of the trace of the extrinsic curvature can written

$$\dot{K} = g^{ij} \dot{\zeta}_{ij} - 2N \zeta^{ij} \zeta_{ij} - 2\zeta^{ij} D_{(i} N_{j)}. \quad (7.10)$$

The time derivative of the spatial Ricci tensor \dot{R}_{ij} can also be written in terms of ζ_{ij} by using the relation

$$\dot{g}_{ij} = 2N\zeta_{ij} + 2D_{(i}N_{j)}. \quad (7.11)$$

Then the action (3.29) can be replaced by one that depends on ζ_{ij} , $\dot{\zeta}_{ij}$ and g_{ij} . Still the resulting action is more complicated than the one based on the choice (4.1) which is used in the Hamiltonian formalism developed in Sec. 6.

8 Conclusion

In this paper we have analyzed a new version of the so called covariant renormalizable gravity [17] as an example of higher derivative theory of gravity which involves spontaneous Lorentz and/or diffeomorphism symmetry breaking. It was earlier shown that this theory possesses the correct number of degrees of freedom when fluctuations around the flat background are analyzed and also that these fluctuations have modified dispersion relations so that the theory is power-counting renormalizable [17]. This is a remarkable feature, because the theory contains higher time and space derivatives which could imply potential problems with ghosts. For that reason it would certainly be nice to analyze the fully nonlinear theory with manifest diffeomorphism invariance. However, due to the complexity of the action we restricted our analysis to the case of the theory where the equation of motion (2.3) is solved for the scalar field ϕ . Using a foliation of spacetime defined by ϕ we obtained the ADM representation of the action of CRG and its Hamiltonian structure. It turned out that the lapse function N has to depend on time only. As a result the theory obeys the projectability condition on N and lacks a local Hamiltonian constraint, which suggests that it is not possible to eliminate the additional scalar mode. Furthermore the fact that the CRG action involves higher time derivatives implies that it contains another additional degree of freedom, which can be seen when we introduce auxiliary fields. In summary, the theory contains 4 gravitational degrees of freedom. 3 of the modes are carried by the spatial metric in conjunction with an auxiliary field ξ . Another extra scalar mode is carried by the trace component ζ of the higher-order variable (4.1) which contains the second-order time derivatives present in the ADM representation of the action (3.29). We argued that the theory contains a degree of freedom that carries negative energy, which will destabilize the theory due to interactions with positive energy degrees of freedom. Apparently the available constraints are unable to prevent this pathology. As a result even “empty space” can decay into compensating positive and negative energy excitations. Then we conjectured that generally covariant higher derivative theories of gravity which involve spontaneous (constraint induced) Lorentz and/or diffeomorphism symmetry breaking will in general share this problem with CRG. This is motivated by the difficulties in cancelling time derivatives from covariant higher-order theory and the reduced structure of constraints due to symmetry breaking.

We should note that the secondary constraints of the theory are very complicated. Especially the second-class secondary constraints turned out inconveniently complicated. As a result the Hamiltonian has a very complicated form. This will make the canonical quantization of the theory a very hard task. Covariant quantization would likely be a feasible approach.

Very recently we have begin to suspect that in this type of theory it might be better to treat the normal n^μ as a genuine dynamical variable with constraints restricting it to unit norm and zero vorticity. This way a more general treatment could perhaps be achieved.

In covariant form the action would be defined as

$$S_3 = \int d^4x \sqrt{-^{(4)}g} \left[\frac{^{(4)}R}{2\kappa^2} - \alpha (R^{\alpha\beta} + KK^{\alpha\beta} - D^\alpha a^\beta) \right. \\ \left. \times (n^\mu n^\nu \nabla_\mu \nabla_\nu + \nabla^\mu \nabla_\mu) (R_{\alpha\beta} + KK_{\alpha\beta} - D_\alpha a_\beta) \right. \\ \left. + \lambda (n_\mu n^\mu + 1) + B^{\mu\nu} \mathcal{F}_{\mu\nu} + M_{\mu\nu\rho\sigma} B^{\mu\nu} B^{\rho\sigma} \right], \quad (8.1)$$

where the vorticity for n_μ is

$$\mathcal{F}_{\mu\nu} = g^\rho{}_\mu g^\sigma{}_\nu \nabla_{[\rho} n_{\sigma]} \quad (8.2)$$

and λ , $B^{\mu\nu}$ and $M_{\mu\nu\rho\sigma}$ are Lagrange multiplier fields. Variations of the action with respect to the Lagrange multipliers yields

$$n_\mu n^\mu = -1, \quad B^{\mu\nu} = 0, \quad \mathcal{F}_{\mu\nu} = 0. \quad (8.3)$$

The normal would associated with the scalar field by

$$n_\mu = -N \nabla_\mu \phi \quad (8.4)$$

and choosing it to be the time $\phi = t$ (ADM gauge choice) yields the ADM formulation of the theory. This approach would be similar to the covariant formulation of HL gravity [45] (also see [22, 13]), but with a much more complicated action: higher-order time derivatives and several extra kinetic terms etc. Indeed, the only conceivable advantage of CRG over HL gravity is the spontaneous breaking of Lorentz invariance in the high energy regime. The idea of achieving renormalizability of gravity via spontaneous symmetry breaking is certainly appealing, but we doubt whether it is worth the price of accepting such a complicated Hamiltonian structure, let alone a pathological extra degree of freedom which is simply unacceptable.

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A Appendix

A.1 Symmetrization

Symmetrization and antisymmetrization of tensor indices is denoted by parentheses and brackets, respectively, e.g.

$$B_{(\alpha\beta)} = \frac{1}{2} (B_{\alpha\beta} + B_{\beta\alpha}), \quad B_{[\alpha\beta]} = \frac{1}{2} (B_{\alpha\beta} - B_{\beta\alpha}).$$

We may also use the following notation if it is more convenient

$$B_{\alpha\beta} + (\alpha \leftrightarrow \beta) = 2B_{(\alpha\beta)}, \quad B_{\alpha\beta} - (\alpha \leftrightarrow \beta) = 2B_{[\alpha\beta]},$$

as can be the case in long expressions with many terms.

A.2 Decomposition of the Riemann tensor

Decomposition of the Riemann tensor of spacetime into components tangent and normal to the spatial hypersurfaces Σ_t is given by the following identities.

1. Gauss relation

$$g^\gamma_\mu g^\nu_\delta g^\rho_\alpha g^\sigma_\beta {}^{(4)}R^\mu_{\nu\rho\sigma} = R^\gamma_{\delta\alpha\beta} + K^\gamma_\alpha K_{\delta\beta} - K^\gamma_\beta K_{\alpha\delta} \quad (\text{A.1})$$

2. Codazzi relation

$$g^\gamma_\mu n^\nu g^\rho_\alpha g^\sigma_\beta {}^{(4)}R^\mu_{\nu\rho\sigma} = 2D_{[\alpha} K^\gamma_{\beta]} \quad (\text{A.2})$$

3. Ricci equation

$$g_{\alpha\mu} n^\nu g^\rho_\beta n^\sigma {}^{(4)}R^\mu_{\nu\rho\sigma} = K_{\alpha\mu} K^\mu_\beta + \frac{1}{N} D_\alpha D_\beta N - \frac{1}{N} \mathcal{L}_{Nn} K_{\alpha\beta} \quad (\text{A.3})$$

These imply for example the decomposition of the scalar curvature of spacetime

$${}^{(4)}R = R + K_{ij} K^{ij} - K^2 + 2\nabla_\mu (n^\mu K) - \frac{2}{N} D^i D_i N, \quad (\text{A.4})$$

where

$$2\nabla_\mu (n^\mu K) = 2K^2 + \frac{2}{N} (\dot{K} - N^i \partial_i K).$$

A.3 Decomposition of covariant derivatives of tensors tangent to spatial hypersurfaces

Here we shall obtain space-time decompositions for first and second-order covariant derivatives of tensors tangent to spatial hypersurfaces Σ_t .

In the following calculations we will frequently need the covariant derivative of the orthogonal projector onto the spatial hypersurface

$$\nabla_\mu g^\nu_\alpha = (K_\mu^\nu - n_\mu a^\nu) n_\alpha + n^\nu (K_{\mu\alpha} - n_\mu a_\alpha). \quad (\text{A.5})$$

Consider a symmetric rank 2 covariant tensor field $A_{\alpha\beta}$ that is tangent to Σ_t , such as the two tensor fields inside the parentheses in (3.24). Such a tensor field is invariant under the orthogonal projector, $A_{\alpha\beta} = g^\mu_\alpha g^\nu_\beta A_{\mu\nu}$, since $n^\alpha A_{\alpha\beta} = n^\alpha A_{\beta\alpha} = 0$. Let us decompose the covariant derivative of $A_{\alpha\beta}$:

$$\begin{aligned} \nabla_\mu A_{\alpha\beta} &= \delta^\nu_\mu \nabla_\nu (g^\rho_\alpha g^\sigma_\beta A_{\rho\sigma}) = (g^\nu_\mu - n^\nu n_\mu) g^\rho_\alpha g^\sigma_\beta \nabla_\nu A_{\rho\sigma} + \nabla_\mu (g^\rho_\alpha g^\sigma_\beta) A_{\rho\sigma} \\ &= g^\nu_\mu g^\rho_\alpha g^\sigma_\beta \nabla_\nu A_{\rho\sigma} - n_\mu g^\rho_\alpha g^\sigma_\beta \nabla_n A_{\rho\sigma} + (\nabla_\mu g^\rho_\alpha g^\sigma_\beta + g^\rho_\alpha \nabla_\mu g^\sigma_\beta) A_{\rho\sigma} \\ &= D_\mu A_{\alpha\beta} - n_\mu \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}^\nu A_{\beta)\nu} \right) + 2(K_\mu^\nu - n_\mu a^\nu) n_{(\alpha} A_{\beta)\nu} \\ &= D_\mu A_{\alpha\beta} - n_\mu \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}^\nu A_{\beta)\nu} + 2n_{(\alpha} A_{\beta)\nu} a^\nu \right) + 2K_\mu^\nu n_{(\alpha} A_{\beta)\nu} \end{aligned} \quad (\text{A.6})$$

The component of $\nabla_\mu A_{\alpha\beta}$ that is fully tangent to Σ_t is simply $g^\nu_\mu g^\rho_\alpha g^\sigma_\beta \nabla_\nu A_{\rho\sigma} = D_\mu A_{\alpha\beta}$. For completeness let us list the rest 7 components of the decomposition of $\nabla_\mu A_{\alpha\beta}$:

$$\begin{array}{ll}
-n_\mu & n^\nu g^\rho_\alpha g^\sigma_\beta \nabla_\nu A_{\rho\sigma} = \frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\nu A_{\beta)\nu}, \\
-n_\alpha & g^\nu_\mu n^\rho g^\sigma_\beta \nabla_\nu A_{\rho\sigma} = -K_\mu{}^\nu A_{\nu\beta}, \\
-n_\beta & g^\nu_\mu g^\rho_\alpha n^\sigma \nabla_\nu A_{\rho\sigma} = -K_\mu{}^\nu A_{\alpha\nu}, \\
+n_\mu n_\alpha & n^\nu n^\rho g^\sigma_\beta \nabla_\nu A_{\rho\sigma} = -a^\nu A_{\nu\beta}, \\
+n_\mu n_\beta & n^\nu g^\rho_\alpha n^\sigma \nabla_\nu A_{\rho\sigma} = -a^\nu A_{\alpha\nu}, \\
+n_\alpha n_\beta & g^\nu_\mu n^\rho n^\sigma \nabla_\nu A_{\rho\sigma} = 0, \\
-n_\mu n_\alpha n_\beta & n^\nu n^\rho n^\sigma \nabla_\nu A_{\rho\sigma} = 0,
\end{array}$$

where in the left-hand side column we present the part of each component that are normal to Σ_t in the decomposition (A.6).

For future needs we generalize Eqs. (3.18), (3.19) and (A.6) for a generic covariant tensor field $B_{\alpha_1 \dots \alpha_k}$ that is tangent to Σ_t :

$$\frac{1}{N} \mathcal{L}_{Nn} B_{\alpha_1 \dots \alpha_k} = \nabla_n B_{\alpha_1 \dots \alpha_k} + (K_{\alpha_1}{}^\mu - n_{\alpha_1} a^\mu) B_{\mu \dots \alpha_k} + \dots + (K_{\alpha_k}{}^\mu - n_{\alpha_k} a^\mu) B_{\alpha_1 \dots \mu}, \quad (\text{A.7})$$

$$\frac{1}{N} \mathcal{L}_{Nn} B_{\alpha_1 \dots \alpha_k} = g^{\mu_1}{}_{\alpha_1} \dots g^{\mu_k}{}_{\alpha_k} \nabla_n B_{\mu_1 \dots \mu_k} + K_{\alpha_1}{}^\mu B_{\mu \dots \alpha_k} + \dots + K_{\alpha_k}{}^\mu B_{\alpha_1 \dots \mu}, \quad (\text{A.8})$$

$$\begin{aligned}
\nabla_\mu B_{\alpha_1 \dots \alpha_k} &= D_\mu B_{\alpha_1 \dots \alpha_k} - n_\mu \left(\frac{1}{N} \mathcal{L}_{Nn} B_{\alpha_1 \dots \alpha_k} - K_{\alpha_1}{}^\nu B_{\nu \dots \alpha_k} - \dots - K_{\alpha_k}{}^\nu B_{\alpha_1 \dots \nu} \right. \\
&\quad \left. + n_{\alpha_1} a^\nu B_{\nu \dots \alpha_k} + \dots + n_{\alpha_k} a^\nu B_{\alpha_1 \dots \nu} \right) + n_{\alpha_1} K_\mu{}^\nu B_{\nu \dots \alpha_k} + \dots + n_{\alpha_k} K_\mu{}^\nu B_{\alpha_1 \dots \nu}.
\end{aligned} \quad (\text{A.9})$$

It is worth to note that due to the implication $a^\mu = 0$ of the condition $N = N(t)$ [see (3.22)], in (A.7) $\nabla_n B_{\alpha_1 \dots \alpha_k}$ is already tangent to Σ_t , and hence the projection of (A.7) in (A.8) is trivial. However, as we did in (3.18) and (A.6), we continue to write a^μ explicitly in the decompositions of the covariant derivatives, in order to retain generality in this point.

Then consider the second order covariant derivative of $A_{\alpha\beta}$. We decompose the second order covariant derivative of $A_{\alpha\beta}$ similarly as the first-order covariant derivative in (A.6):

$$\begin{aligned}
\nabla_\mu \nabla_\nu A_{\alpha\beta} &= \delta_\mu^\rho \delta_\nu^\sigma \nabla_\rho \nabla_\sigma (g^\lambda_\alpha g^\tau_\beta A_{\lambda\tau}) \\
&= (g^\rho_\mu - n^\rho n_\mu) (g^\sigma_\nu - n^\sigma n_\nu) g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} \\
&\quad + 2\nabla_{(\mu} (g^\lambda_\alpha g^\tau_\beta) \nabla_{\nu)} A_{\lambda\tau} + \nabla_\mu \nabla_\nu (g^\lambda_\alpha g^\tau_\beta) A_{\lambda\tau} \\
&= g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} - n_\mu n^\rho g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} \\
&\quad - n_\nu g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} + n_\mu n_\nu n^\rho n^\sigma g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} \\
&\quad + 2\nabla_{(\mu} (g^\rho_\alpha g^\sigma_\beta) \nabla_{\nu)} A_{\rho\sigma} + \nabla_\mu \nabla_\nu (g^\rho_\alpha g^\sigma_\beta) A_{\rho\sigma}.
\end{aligned} \quad (\text{A.10})$$

However, now we still have to decompose all the terms in (A.10). In order to obtain the component of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ that is fully tangent to Σ_t we use (3.25) to calculate

$$\begin{aligned}
D_\mu D_\nu A_{\alpha\beta} &= g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho D_\sigma A_{\lambda\tau} = g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho (g^\phi_\sigma g^\varphi_\lambda g^\chi_\tau \nabla_\phi A_{\varphi\chi}) \\
&= g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho (g^\phi_\sigma g^\varphi_\lambda g^\chi_\tau) \nabla_\phi A_{\varphi\chi} + g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau},
\end{aligned}$$

which together with (A.6) finally gives

$$g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} = D_\mu D_\nu A_{\alpha\beta} - K_{\mu\nu} \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\rho A_{\beta)\rho} \right) + 2K_{\mu(\alpha} K_{\nu}{}^\rho A_{\rho|\beta)}. \quad (\text{A.11})$$

Next consider the $-n_\mu$ component of the decomposition of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ [the second term in (A.10)]. It can be decomposed by using (3.25) to write

$$\nabla_n D_\nu A_{\alpha\beta} = \nabla_n (g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\sigma A_{\lambda\tau}) = \nabla_n (g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta) \nabla_\sigma A_{\lambda\tau} + n^\rho g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau}, \quad (\text{A.12})$$

where the last term is what we are looking to decompose. Decomposition of the left-hand side of (A.12) is given by (A.7)

$$\begin{aligned} \nabla_n D_\nu A_{\alpha\beta} &= \frac{1}{N} \mathcal{L}_{Nn} D_\nu A_{\alpha\beta} - (K_\nu{}^\rho - n_\nu a^\rho) D_\rho A_{\alpha\beta} \\ &\quad - (K_\alpha{}^\rho - n_\alpha a^\rho) D_\nu A_{\rho\beta} - (K_\beta{}^\rho - n_\beta a^\rho) D_\nu A_{\alpha\rho}. \end{aligned}$$

Decomposition of the first term in the right-hand side of (A.12) is given by (A.6) and taking the derivative of the projectors,

$$\begin{aligned} \nabla_n (g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta) \nabla_\sigma A_{\lambda\tau} &= n_\nu a^\rho D_\rho A_{\alpha\beta} + a_\nu \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\rho A_{\beta)\rho} \right) \\ &\quad + 2n_{(\alpha} D_{\nu} A_{|\beta)\rho} a^\rho - 2K_\nu{}^\rho a_{(\alpha} A_{\beta)\rho}. \end{aligned}$$

Thus we obtain the $-n_\mu$ component of the decomposition (A.10) of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$

$$\begin{aligned} n^\rho g^\sigma_\nu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} &= \frac{1}{N} \mathcal{L}_{Nn} D_\nu A_{\alpha\beta} - K_\nu{}^\rho D_\rho A_{\alpha\beta} - 2K_{(\alpha}{}^\rho D_\nu A_{|\beta)\rho} \\ &\quad - a_\nu \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\rho A_{\beta)\rho} \right) + 2K_\nu{}^\rho a_{(\alpha} A_{\beta)\rho}. \end{aligned} \quad (\text{A.13})$$

The $-n_\nu$ component of the decomposition of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ [the third term in (A.10)] can be written

$$g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} = n^\rho g^\sigma_\mu g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} + g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta [\nabla_\rho, \nabla_\sigma] A_{\lambda\tau}. \quad (\text{A.14})$$

In the second term we use the *Ricci identity* and the fact that $A_{\alpha\beta}$ is tangent to Σ_t to obtain

$$\begin{aligned} g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta [\nabla_\rho, \nabla_\sigma] A_{\lambda\tau} &= -g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta \left({}^{(4)}R^\phi_{\lambda\rho\sigma} A_{\phi\tau} + {}^{(4)}R^\phi_{\tau\rho\sigma} A_{\lambda\phi} \right) \\ &= -g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta g^{\phi\varphi} \left({}^{(4)}R_{\varphi\lambda\rho\sigma} A_{\phi\tau} + {}^{(4)}R_{\varphi\tau\rho\sigma} A_{\lambda\phi} \right) \\ &= -g_{\mu\rho} n^\sigma g^\varphi_\phi \left(g^\lambda_\alpha {}^{(4)}R^\rho_{\sigma\varphi\lambda} A^\phi_\beta + g^\tau_\beta {}^{(4)}R^\rho_{\sigma\varphi\tau} A_\alpha{}^\phi \right) \\ &= -2D_{[\rho} K_{\mu|\alpha]} A^\rho_{\beta]} - 2D_{[\rho} K_{\mu|\beta]} A_\alpha{}^\rho \\ &= 2D_{(\alpha} K_{\mu\rho} A^\rho_{|\beta)} - 2D_\rho K_{\mu(\alpha} A^\rho_{\beta)}. \end{aligned}$$

where we have used the Codazzi relation (A.2) in the second last equality. Thus the $-n_\nu$ component of the decomposition (A.10) of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ is

$$\begin{aligned} g^\rho_\mu n^\sigma g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} &= \frac{1}{N} \mathcal{L}_{Nn} D_\mu A_{\alpha\beta} - K_\mu{}^\rho D_\rho A_{\alpha\beta} - 2K_{(\alpha}{}^\rho D_\mu A_{|\beta)\rho} \\ &\quad - a_\mu \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\rho A_{\beta)\rho} \right) + 2K_\mu{}^\rho a_{(\alpha} A_{\beta)\rho} \\ &\quad + 2D_{(\alpha} K_{\mu\rho} A^\rho_{|\beta)} - 2D_\rho K_{\mu(\alpha} A^\rho_{\beta)}. \end{aligned} \quad (\text{A.15})$$

Next consider the $+n_\mu n_\nu$ component of the decomposition of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ [the fourth term in (A.10)]. Let us write

$$\nabla_n (g^\lambda_\alpha g^\tau_\beta \nabla_n A_{\lambda\tau}) = \nabla_n (n^\sigma g^\lambda_\alpha g^\tau_\beta) \nabla_\sigma A_{\lambda\tau} + n^\rho n^\sigma g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau}, \quad (\text{A.16})$$

where the last term is what we are looking to decompose. The left-hand side can alternatively be written by using (A.7) and (A.8)

$$\begin{aligned} \nabla_n (g^\lambda_\alpha g^\tau_\beta \nabla_n A_{\lambda\tau}) &= \nabla_n \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\rho A_{\beta)\rho} \right) \\ &= \frac{1}{N} \mathcal{L}_{Nn} \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} - 2K_{(\alpha}{}^\rho A_{\beta)\rho} \right) \\ &\quad - (K_\alpha{}^\rho - n_\alpha a^\rho) \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} - 2K_{(\rho}{}^\sigma A_{\beta)\sigma} \right) \\ &\quad - \left(K_\beta{}^\rho - n_\beta a^\rho \right) \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\rho} - K_{(\alpha}{}^\sigma A_{\rho)\sigma} \right). \end{aligned}$$

We also need to compute

$$\begin{aligned} \nabla_n (n^\sigma g^\lambda_\alpha g^\tau_\beta) \nabla_\sigma A_{\lambda\tau} &= a^\rho D_\rho A_{\alpha\beta} + n_\alpha a^\rho \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} - 2K_{(\rho}{}^\sigma A_{\beta)\sigma} \right) \\ &\quad + n_\beta a^\rho \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\rho} - K_{(\alpha}{}^\sigma A_{\rho)\sigma} \right) - 2a_{(\alpha} A_{\beta)\rho} a^\rho \end{aligned}$$

Thus we obtain the $+n_\mu n_\nu$ component of the decomposition (A.10) of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$

$$\begin{aligned} n^\rho n^\sigma g^\lambda_\alpha g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} &= \frac{1}{N} \mathcal{L}_{Nn} \left(\frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\beta} \right) - \frac{2}{N} \mathcal{L}_{Nn} K_{(\alpha}{}^\rho A_{\beta)\rho} \\ &\quad - 4K_{(\alpha}{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{|\beta)\rho} + 2K_{(\alpha}{}^\rho K_{\beta)}{}^\sigma A_{\rho\sigma} + 2K_{(\alpha}{}^\rho K_{\rho}{}^\sigma A_{|\beta)\sigma} \\ &\quad - a^\rho D_\rho A_{\alpha\beta} + 2a_{(\alpha} A_{\beta)\rho} a^\rho. \end{aligned} \quad (\text{A.17})$$

The rest 12 components of the decomposition of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ are given in the last two terms of Eq. (A.10). Let us calculate them

$$\begin{aligned} 2\nabla_{(\mu} (g^\rho_\alpha g^\sigma_\beta) \nabla_{\nu)} A_{\rho\sigma} &= n_\alpha \left[K_\mu{}^\rho D_\nu A_{\rho\beta} - n_\mu \left(a^\rho D_\nu A_{\rho\beta} + K_\nu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} \right. \right. \\ &\quad \left. \left. - K_\nu{}^\rho K_\rho{}^\sigma A_{\sigma\beta} - K_\nu{}^\rho K_\beta{}^\sigma A_{\rho\sigma} \right) + n_\mu n_\nu \left(a^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} \right. \right. \\ &\quad \left. \left. - a^\rho K_\rho{}^\sigma A_{\sigma\beta} - a^\rho K_\beta{}^\sigma A_{\rho\sigma} \right) + (\mu \leftrightarrow \nu) \right] \\ &\quad + [(K_\mu{}^\rho - n_\mu a^\rho) (K_{\nu\alpha} - n_\nu a_\alpha) + (\mu \leftrightarrow \nu)] A_{\rho\beta} + (\alpha \leftrightarrow \beta), \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} \nabla_\mu \nabla_\nu (g^\rho_\alpha g^\sigma_\beta) A_{\rho\sigma} &= \nabla_\mu \nabla_\nu g^\rho_\alpha A_{\rho\beta} + \nabla_\mu g^\rho_\alpha \nabla_\nu g^\sigma_\beta A_{\rho\sigma} + (\alpha \leftrightarrow \beta) \\ &= n_\alpha \left[D_\mu K_{\nu\rho} - K_{\mu\nu} a_\rho - n_\mu \left(\frac{1}{N} \mathcal{L}_{Nn} K_{\nu\rho} - 2K_{\nu\sigma} K_\rho{}^\sigma - a_\nu a_\rho \right) \right. \\ &\quad \left. - n_\nu (D_\mu a_\rho - K_\mu{}^\sigma K_{\sigma\rho}) + n_\mu n_\nu \left(\frac{1}{N} \mathcal{L}_{Nn} a_\rho - 2K_{\rho\sigma} a^\sigma \right) \right] A^\rho_\beta \\ &\quad + [(K_\mu{}^\rho - n_\mu a^\rho) (K_{\nu\alpha} - n_\nu a_\alpha) + (\mu \leftrightarrow \nu)] A_{\rho\beta} \\ &\quad + n_\alpha n_\beta (K_\mu{}^\rho - n_\mu a^\rho) (K_\nu{}^\sigma - n_\nu a^\sigma) A_{\rho\sigma} + (\alpha \leftrightarrow \beta). \end{aligned} \quad (\text{A.19})$$

Taking the sum of (A.18) and (A.19) gives

$$\begin{aligned}
& 2\nabla_{(\mu} (g^\rho_\alpha g^\sigma_\beta) \nabla_{|\nu)} A_{\rho\sigma} + \nabla_\mu \nabla_\nu (g^\rho_\alpha g^\sigma_\beta) A_{\rho\sigma} \\
&= n_\alpha \left[K_\mu{}^\rho D_\nu A_{\rho\beta} + K_\nu{}^\rho D_\mu A_{\rho\beta} + D_\mu K_{\nu\rho} A^\rho_\beta - K_{\mu\nu} a^\rho A_{\rho\beta} \right. \\
&\quad - n_\mu \left(a^\rho D_\nu A_{\rho\beta} + K_\nu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} + \frac{1}{N} \mathcal{L}_{Nn} K_{\nu\rho} A^\rho_\beta \right. \\
&\quad \left. \left. - 3K_\nu{}^\rho K_\rho{}^\sigma A_{\sigma\beta} - K_\nu{}^\rho K_\beta{}^\sigma A_{\rho\sigma} - a_\nu a^\rho A_{\rho\beta} \right) \right. \\
&\quad \left. - n_\nu \left(a^\rho D_\mu A_{\rho\beta} + D_\mu a_\rho A^\rho_\beta + K_\mu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} \right. \right. \\
&\quad \left. \left. - 2K_\mu{}^\rho K_\rho{}^\sigma A_{\sigma\beta} - K_\mu{}^\rho K_\beta{}^\sigma A_{\rho\sigma} \right) \right. \\
&\quad \left. + n_\mu n_\nu 2 \left(a^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} + \frac{1}{2N} \mathcal{L}_{Nn} a_\rho A^\rho_\beta - 2a^\rho K_\rho{}^\sigma A_{\sigma\beta} \right. \right. \\
&\quad \left. \left. - a^\rho K_\beta{}^\sigma A_{\rho\sigma} \right) \right] + n_\alpha n_\beta (K_\mu{}^\rho - n_\mu a^\rho) (K_\nu{}^\sigma - n_\nu a^\sigma) A_{\rho\sigma} \\
&\quad + (\alpha \leftrightarrow \beta). \tag{A.20}
\end{aligned}$$

Now we can read the rest of the components of the decomposition of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$ from (A.20):

$$\begin{aligned}
& -n_\alpha \quad g^\rho_\mu g^\sigma_\nu n^\lambda g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -2K_\mu{}^\rho D_\nu A_{\rho\beta} - D_\mu K_{\nu\rho} A^\rho_\beta + K_{\mu\nu} a^\rho A_{\rho\beta}, \\
& -n_\beta \quad g^\rho_\mu g^\sigma_\nu g^\lambda_\alpha n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -2K_\mu{}^\rho D_\nu A_{\alpha\rho} - D_\mu K_{\nu\rho} A^\rho_\alpha + K_{\mu\nu} A_{\alpha\rho} a^\rho, \\
& +n_\mu n_\alpha \quad n^\rho g^\sigma_\nu n^\lambda g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -K_\nu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} - \frac{1}{N} \mathcal{L}_{Nn} K_{\nu\rho} A^\rho_\beta - a^\rho D_\nu A_{\rho\beta} \\
& \quad + 3K_\nu{}^\rho K_\rho{}^\sigma A_{\sigma\beta} + K_\nu{}^\rho K_\beta{}^\sigma A_{\rho\sigma} + a_\nu a^\rho A_{\rho\beta}, \\
& +n_\mu n_\beta \quad n^\rho g^\sigma_\nu g^\lambda_\alpha n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -K_\nu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\rho} - \frac{1}{N} \mathcal{L}_{Nn} K_{\nu\rho} A^\rho_\alpha - a^\rho D_\nu A_{\alpha\rho} \\
& \quad + 3K_\nu{}^\rho K_\rho{}^\sigma A_{\alpha\sigma} + K_\nu{}^\rho K_\alpha{}^\sigma A_{\rho\sigma} + a_\nu a^\rho A_{\alpha\rho}, \\
& +n_\nu n_\alpha \quad g^\rho_\mu n^\sigma n^\lambda g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -K_\mu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} - a^\rho D_\mu A_{\rho\beta} - D_\mu a_\rho A^\rho_\beta \\
& \quad + 2K_\mu{}^\rho K_\rho{}^\sigma A_{\sigma\beta} + K_\mu{}^\rho K_\beta{}^\sigma A_{\rho\sigma}, \\
& +n_\nu n_\beta \quad g^\rho_\mu n^\sigma g^\lambda_\alpha n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -K_\mu{}^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\rho} - a^\rho D_\mu A_{\alpha\rho} - D_\mu a_\rho A^\rho_\alpha \\
& \quad + 2K_\mu{}^\rho K_\rho{}^\sigma A_{\alpha\sigma} + K_\mu{}^\rho K_\alpha{}^\sigma A_{\rho\sigma}, \\
& +n_\alpha n_\beta \quad g^\rho_\mu g^\sigma_\nu n^\lambda n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = 2K_\mu{}^\rho K_\nu{}^\sigma A_{\rho\sigma}, \\
& -n_\mu n_\nu n_\alpha \quad n^\rho n^\sigma n^\lambda g^\tau_\beta \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -2a^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\rho\beta} - \frac{1}{N} \mathcal{L}_{Nn} a_\rho A^\rho_\beta \\
& \quad + 4a^\rho K_\rho{}^\sigma A_{\sigma\beta} + 2a^\rho K_\beta{}^\sigma A_{\rho\sigma}, \\
& -n_\mu n_\nu n_\beta \quad n^\rho n^\sigma g^\lambda_\alpha n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = -2a^\rho \frac{1}{N} \mathcal{L}_{Nn} A_{\alpha\rho} - \frac{1}{N} \mathcal{L}_{Nn} a_\rho A^\rho_\alpha \\
& \quad + 4a^\rho K_\rho{}^\sigma A_{\alpha\sigma} + 2a^\rho K_\alpha{}^\sigma A_{\rho\sigma}, \\
& -n_\mu n_\alpha n_\beta \quad n^\rho g^\sigma_\nu n^\lambda n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = 2a^\rho K_\nu{}^\sigma A_{\rho\sigma}, \\
& -n_\nu n_\alpha n_\beta \quad g^\rho_\mu n^\sigma n^\lambda n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = 2K_\mu{}^\rho a^\sigma A_{\rho\sigma}, \\
& +n_\mu n_\nu n_\alpha n_\beta \quad n^\rho n^\sigma n^\lambda n^\tau \nabla_\rho \nabla_\sigma A_{\lambda\tau} = 2a^\rho a^\sigma A_{\rho\sigma}.
\end{aligned}$$

Finally we have obtained every component of the decomposition of $\nabla_\mu \nabla_\nu A_{\alpha\beta}$.

One thing remains to be uncovered in the space-time decomposition. The Lie derivative (A.7) contains a time derivative of $B_{\alpha_1 \dots \alpha_k}$ that needs to be uncovered. For this purpose we assume a coordinate system on Σ_t and use the connection coefficients explicitly. We denote the connection coefficients of ∇ and D with

$${}^{(4)}\Gamma_{\nu\rho}^\mu = \frac{1}{2} {}^{(4)}g^{\mu\sigma} (\partial_\nu {}^{(4)}g_{\sigma\rho} + \partial_\rho {}^{(4)}g_{\nu\sigma} - \partial_\sigma {}^{(4)}g_{\nu\rho}) \quad (\text{A.21})$$

and

$$\Gamma_{jk}^i = \frac{1}{2} g^{im} (\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk}) , \quad (\text{A.22})$$

respectively. Using (3.3) and (3.4) we obtain

$${}^{(4)}\Gamma_{00}^0 = \frac{1}{N} \left(\dot{N} + N^i \partial_i N + K_{ij} N^i N^j \right) , \quad (\text{A.23})$$

$${}^{(4)}\Gamma_{0i}^0 = \frac{1}{N} (\partial_i N + K_{ij} N^j) , \quad (\text{A.24})$$

$${}^{(4)}\Gamma_{ij}^0 = \frac{1}{N} K_{ij} , \quad (\text{A.25})$$

$${}^{(4)}\Gamma_{00}^i = g^{ij} \left(N \partial_j N + \dot{N}_j - D_j N_k N^k \right) - \frac{N^i}{N} \left(\dot{N} + N^j \partial_j N + K_{jk} N^j N^k \right) , \quad (\text{A.26})$$

$${}^{(4)}\Gamma_{0j}^i = g^{ik} (N K_{kj} + D_j N_k) - \frac{N^i}{N} (\partial_j N + N^k K_{kj}) , \quad (\text{A.27})$$

$${}^{(4)}\Gamma_{jk}^i = \Gamma_{jk}^i - \frac{N^i K_{jk}}{N} . \quad (\text{A.28})$$

Then we obtain the covariant derivative in (A.7) by using the connection coefficients. For our purposes it is sufficient to consider only the spatial components of (A.7) because the action (4.2) involves only them (4.3). We obtain

$$\begin{aligned} \nabla_n B_{i_1 \dots i_k} &= n^\mu (\partial_\mu B_{i_1 \dots i_k} - {}^{(4)}\Gamma_{\mu i_1}^\nu B_{\nu \dots i_k} - \dots - {}^{(4)}\Gamma_{\mu i_k}^\nu B_{i_1 \dots \nu}) \\ &= \frac{1}{N} \left[\dot{B}_{i_1 \dots i_k} - N^j D_j B_{i_1 \dots i_k} - (N K_{i_1}^j + D_{i_1} N^j) B_{j \dots i_k} \right. \\ &\quad \left. - \dots - (N K_{i_k}^j + D_{i_k} N^j) B_{i_1 \dots j} \right] \end{aligned} \quad (\text{A.29})$$

Thus we can write the spatial components of (A.7) as

$$\begin{aligned} \frac{1}{N} \mathcal{L}_N B_{i_1 \dots i_k} &= \frac{1}{N} \left(\dot{B}_{i_1 \dots i_k} - N^j D_j B_{i_1 \dots i_k} - D_{i_1} N^j B_{j \dots i_k} - \dots - D_{i_k} N^j B_{i_1 \dots j} \right) \\ &= \frac{1}{N} \left(\dot{B}_{i_1 \dots i_k} - \mathcal{L}_N B_{i_1 \dots i_k} \right) , \end{aligned} \quad (\text{A.30})$$

where \mathcal{L}_N denotes the Lie derivative along the shift vector N^i in Σ_t .

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